The asset pricing in finite and discrete time-horizon

Medvegyev Péter*

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Abstract

In the article we shortly discuss the proof of the theorem of Dalang–Morton–Willinger. We show that the proof of the theorem depends on some interesting general properties of the stochastic convergence. We also present a proof for the so-called second fundamental theorem of asset pricing.

Contents

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1 The first fundamental theorem of asset pricing

The Dalang–Morton–Willinger theorem[1] is one of the most striking theorems of Mathematical Finance. By this theorem on a finite and discrete time-horizon the so-called no-arbitrage condition is necessary and sufficient for the existence of an equivalent martingale measure. The theorem is noticeable from many point of view. First of all it is very simple to understand and remember and it is also a very general theorem, therefore it is a very elegant statement. The generality of the theorem comes from the fact, that the only restriction of the theorem is the assumed finiteness of the time-horizon. An important fact about the proof of the theorem is that it is nearly the same as the proof of a much simpler theorem, the so-called Harrison–Pliska theorem[2], which basically says the same thing, with a notable difference, that it assumes that the probability space describing the possible outcomes is finite. The proof of the Harrison–Pliska theorem is a simple exercise about the duality theorem of linear programming. The difference between the proofs of the two theorems comes from the conditions of these results. The proof of the Dalang–Morton–Willinger theorem, by the nature of the conditions of the theorem, based on measure theory and duality theorems of functional analysis. We will see that because of some remarkable properties of the so-called L^0-space the idea of the proof of the Dalang–Morton–Willinger theorem is very similar to the proof of the much simpler theorem of Harrison and Pliska.

1.1 The Dalang–Morton–Willinger theorem

We assume that the reader is familiar with the basic concepts of the Mathematical Finance, therefore we are giving here only a short summary. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be an arbitrary probability space and let $\mathcal{F} = (\mathcal{F}_t)_{t=0}^{T}$ be a discrete-time filtration. Let $S \mathrel{\hat=} (S(t))_{t=0}^{T}$ be an $\mathcal{F}$-adapted m-dimensional process. As it is well-known adaptedness means that the vector of random variables $S(t)$ is $\mathcal{F}_t$-measurable for every t. We will call S the price process. Let us introduce the linear subspace of $L^0(\Omega, \mathcal{F}_T, \mathbf{P})$

$$K \mathrel{\hat=} \left\{ H : H = \sum_{t=1}^{T} (S(t) - S(t-1), \theta(t)) \right\}$$

where θ runs through the whole set of predictable m-dimensional processes. Recall that a process θ is predictable if $\theta(t)$ is $\mathcal{F}_{t-1}$-measurable for every t. The linear space K is the set of all possible gains which one can win trying to use the changing of the prices of stocks represented by S. As usual L^0_+ denotes the set of non-negative random variables. Let us also introduce the set

$$C \mathrel{\hat=} K - L^0_+,$$

[1]See: [1], [3], [5].
[2]See: [8].

and its closure $\mathrm{cl}(C)$. We use the topology of convergence in probability among the random variables, so $\mathrm{cl}(C)$ is the closure of C in the metric space generated by the stochastic convergence. On discrete and finite time-horizon the most general no-arbitrage theorem is the following:

Theorem 1 (Dalang–Morton–Willinger) *The next statements are equivalent:*

1. $C \cap L^0_+ = \{0\}$.
2. $C \cap L^0_+ = \{0\}$ *and* $C = \mathrm{cl}(C)$.
3. $\mathrm{cl}(C) \cap L^0_+ = \{0\}$.
4. *There is a probability measure* $\mathbf{Q}$ *with bounded and positive Radon–Nikodym derivative* $d\mathbf{Q}/d\mathbf{P}$*, such that the coordinates of the* m*-dimensional process* S *are martingales under* $\mathbf{Q}$*.*

Recall that by the first statement there is no such predictable strategy $(\theta(t))_{t=1}^{T}$ that

$$\sum_{t=1}^{T} (S(t) - S(t-1), \theta(t)) \geq 0$$

and on a set of positive measure the inequality is strict. That is by the first statement there is no-arbitrage in the model represented by S.

1.2 Some fundamental properties of the space L^0

Recall that $L^0(\Omega, \mathcal{A}, \mathbf{P})$ denotes the set of random variables measurable with respect to the σ-algebra $\mathcal{A}$. From now on we drop the parameter $(\Omega, \mathcal{A}, \mathbf{P})$ and we will use the simpler notation L^0. The random variables are equivalence classes with respect to the probability measure $\mathbf{P}$. In L^0 we define the topology of convergence in probability. Recall that the topology of stochastic convergence is metrizable so to check the closedness of a set $Z \subseteq L^0$ it is sufficient to show that if a sequence in Z is convergent then the limit of the sequence is also in Z. A crucial property of the convergence in probability is that every convergent sequence contains a sub-sequence which is almost surely convergent. As the probability measure is finite the almost sure convergence implies the convergence in probability. Summarizing[3] these properties one should observe that a set $Z \subseteq L^0$ is closed if and only if the limit of every almost surely convergent sequence from Z is in Z. The next lemma[4] contains the key compactness type property of L^0 :

[3] If the measure is not discrete, there is no such topology that a sequence is convergent if and only if it is almost surely convergent. Nevertheless we can use the almost sure convergence to prove the closedness of sets in the topology of convergence in probability..

[4] See: [10].

Lemma 2 *Let (η_n) be an $\mathbb{R}^m$-valued sequence of measurable functions. If for every outcome the sequence (η_k) is bounded then there is a strictly increasing, measurable sequence of integers (σ_k) such that for every outcome the sequence (η_{σ_k}) is convergent. On the other hand if $\sup_n \|\eta_n\| = \infty$ then there is a strictly increasing, measurable sequence of integers, (σ_k) such that for every outcome $\lim_{k\to\infty} \|\eta_{\sigma_k}\| = \infty$.*

Proof: Observe that by the Bolzano–Weierstrass theorem for every outcome ω one can select a strictly increasing sequence of integers $(\sigma_k(\omega))$ such that the sequence $\left(\eta_{\sigma_k(\omega)}(\omega)\right)$ is convergent. The main point of the lemma is that the sequence of integers σ_k can be selected in a measurable way. Assume first that (η_n) is a scalar-valued sequence. By the assumptions $\eta_\infty \triangleq \liminf_n \eta_n$ exists and it is finite for every outcome. As (η_n) is measurable η_∞ is also measurable. Let $\sigma_0 \triangleq 0$ and let us define the functions

$$\sigma_k \triangleq \inf\left\{n > \sigma_{k-1} : |\eta_n - \eta_\infty| \leq \frac{1}{k}\right\}.$$

One can easily show that σ_k is measurable for every k and $\eta_{\sigma_k} \to \eta_\infty$. Hence in this case the lemma holds. In the multi-dimensional case first we construct a sequence of indexes which makes the first coordinate convergent then from the subsequence generated by this sequence of indexes we choose another subsequence which makes the second coordinate convergent etc.

To prove the second statement it is sufficient to choose the indexes

$$\sigma_k \triangleq \inf\{n > \sigma_{k-1} : \|\eta_n\| \geq k\}.$$

□

A direct consequence of the lemma is that one can generalize the closedness of finite-dimensional sub-spaces to the case when the scalars are measurable functions.

Lemma 3 (Stricker) *Let $f_1, f_2, \ldots, f_m$ be arbitrary $\mathcal{A}$-measurable functions. Assume that $\mathcal{F} \subseteq \mathcal{A}$ and let*

$$L \triangleq \left\{f : f = \sum_{i=1}^{m} f_i\theta_i, \quad \theta_i \in L^0(\Omega, \mathcal{F}, \mathbf{P})\right\}.$$

L is a closed sub-space of $L^0(\Omega, \mathcal{A}, \mathbf{P})$.

Proof: L is obviously a linear space. Let us select a sequence $l_n \in L$ and let us assume that

$$l_n \triangleq \sum_{i=1}^{m} f_i\theta_i^{(n)} \to l_\infty$$

in probability. One should show that $l_\infty \in L$. To show this one can assume that $l_n \to l_\infty$ almost surely. Using a vectorial notation one can write functions l_n as a scalar product

$$l_n \triangleq (X, y_n),$$

where $X \doteq (f_1, f_2, \ldots, f_m)$ and $y_n \doteq \left(\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_m^{(n)}\right)$. Recall that y_n is $\mathcal{F}$-measurable for every n. Observe that the only problem is that the convergence of (l_n) does not imply the convergence of (y_n). On the other hand one should realize that it is sufficient to show that there is a strictly increasing $\mathcal{F}$-measurable sequence of integers (σ_k) such that $y_{\sigma_k} \to y_\infty$. In this case y_∞ is $\mathcal{F}$-measurable and as

$$(X, y_{\sigma_k}) \to (X, y_\infty) = l_\infty$$

l_∞ is in L. By the previous lemma for the existence of the convergent subsequence (y_{σ_k}) it is sufficient to show that one can select the sequences (y_n) in such a way that for every outcome ω the sequence $(y_n(\omega))$ is bounded. Let Ω_1 be the subset of Ω where this does not hold. As (y_k) is $\mathcal{F}$-measurable Ω_1 is $\mathcal{F}$-measurable. To construct the needed subsequence on the set Ω_1 let us divide the equation

$$l_n(\omega) = (X(\omega), y_n(\omega))$$

with $\|y_n(\omega)\|$:

$$\frac{l_n(\omega)}{\|y_n(\omega)\|} = \left(X(\omega), \frac{y_n(\omega)}{\|y_n(\omega)\|}\right).$$

The sequence $(y_n(\omega) / \|y_n(\omega)\|)$ is bounded, so it has a measurably indexed sub-sequence which is convergent. Of course it can happen that after taking the subsequence for some outcomes the subsequence (y_{σ_k}) is bounded. The set of such outcomes is again $\mathcal{F}$-measurable. Let us delete these outcomes from Ω_1. After this step $\|y_{\sigma_n}(\omega)\| \to \infty$ for the outcomes ω in Ω_1 As (l_n) is convergent on Ω_1

$$\frac{l_{\sigma_n}(\omega)}{\|y_{\sigma_n}(\omega)\|} \to 0.$$

As $(y_n / \|y_n\|)$ is bounded there is an $\mathcal{F}$-measurable function u_∞ such that

$$(X(\omega), u_\infty(\omega)) = 0, \quad \omega \in \Omega_1. \tag{1}$$

$u_\infty(\omega) \in \mathbb{R}^m$ is a limit of vectors which are on the sphere of the unit ball, therefore it is not zero for every $\omega \in \Omega_1$. Hence for every $\omega \in \Omega_1$ one coordinate of $u_\infty(\omega)$ is not zero. Using this coordinate and (1) one can express one of the coordinates of

$$X(\omega) \doteq (f_1(\omega), f_2(\omega), \ldots, f_m(\omega))$$

with the other coordinates. The main point is that in this expression[5] the weights are $\mathcal{F}$-measurable. Substituting the expressed coordinate back to the definition

$$l_{\sigma_n}(\omega) = (X(\omega), y_{\sigma_n}(\omega))$$

one can assume that for every $\omega \in \Omega_1$ only $m-1$ coordinates in $y_{\sigma_n}(\omega)$ are not zero. If the new weights (y_{σ_n}) are not bounded on some set $\Omega_2 \subseteq \Omega_1$ then we repeat the argument. After finite number of steps one can assume that

$$l_n = f_i \theta_i^{(n)}.$$

[5]The coordinate can depend on the outcome.

for some i. In this case if $\theta_i^{(n)}(\omega)$ is not bounded then $f_i(\omega)$ is zero. If Ω_m denotes the $\mathcal{F}$-measurable set where $\left(\theta_i^{(n)}\right)$ is not bounded then taking the sequence $\left(\theta_i^{(n)}\chi_{\Omega_m^c}\right)$ the sequence (y_n) remains $\mathcal{F}$-measurable and bounded. As one can finish the procedure in finite number of steps one can assume that (y_n) is bounded which implies that L is closed.

□

Now we generalize the lemma to the multi-period setup:

Lemma 4 (Kabanov–Stricker) *Let $(X(t))_{t=1}^{T}$ be and arbitrary m-dimensional adapted process and let*

$$L \doteq \left\{ f : f = \sum_{t=1}^{T} \sum_{i=1}^{m} X_i(t)\,\theta_i(t) \right\} = \left\{ f : f = \sum_{t=1}^{T} (X(t), \theta(t)) \right\},$$

where $(\theta(t))_{t=1}^{T}$ runs through all the m-dimensional predictable processes. L is a closed sub-space of $L^0(\mathcal{A}, \mathbf{P})$.

Proof: We prove the theorem by induction with respect to the length of the time-period T. If $T = 1$ then by the previous lemma L is closed. Assume that if the number of time-periods is $T - 1$ then the lemma holds. Let

$$f_n \doteq \sum_{t=1}^{T} (X(t), \theta_n(t)) \to f_\infty.$$

Let

$$\begin{aligned} b_n &\doteq (X(1), \theta_n(1)), \\ c_n &\doteq f_n - b_n \end{aligned}$$

Observe that the convergence of (f_n) does not imply the convergence of (b_n) and (c_n). If the sequence $(\theta_n(1))$ were bounded then taking a subsequence we could assume that $(\theta_n(1))$ is convergent. As one can select the indexes of the subsequence in an $\mathcal{F}_0$-measurable way after taking the subsequences the strategies $(\theta_{\sigma_k}(t))_{t=2}^{T}$ remain predictable. If (b_n) is convergent then (c_n) is also convergent. By the induction hypothesis the limit c_∞ of (c_n) has a representation

$$c_\infty = \sum_{t=2}^{T} (X(t), \theta_\infty(t))$$

therefore in this case the lemma holds. If $(\theta_n(1))$ is not bounded then proceeding as in the previous lemma on some $\mathcal{F}_0$-measurable subset of Ω let

$$\frac{f_n}{\|\theta_n(1)\|} \doteq \left(X(1), \frac{\theta_n(1)}{\|\theta_n(1)\|} \right) + \sum_{t=2}^{T} \left(X(t), \frac{\theta_n(t)}{\|\theta_n(1)\|} \right).$$

By the first lemma one can assume that

$$\left(X(1), \frac{\theta_n(1)}{\|\theta_n(1)\|}\right)$$

is convergent, therefore

$$\sum_{t=2}^{T}\left(X(t), \frac{\theta_n(t)}{\|\theta_n(1)\|}\right)$$

is also convergent. Using again that for $T-1$ time-period the lemma holds the limit c^* has a representation

$$c^* = \sum_{t=2}^{T}(X(t), \theta^*(t)),$$

where of course $(\theta^*(t))_{t=2}^{T}$ is predictable. Hence

$$(X(1), \theta^*(1)) + \sum_{t=2}^{T}(X(t), \theta^*(t)) = 0.$$

Let us take the first time-period where $\theta^*(t) \neq 0$. As in the previous lemma we can eliminate some of the coordinates of $X(t)$ in this time-period in an $\mathcal{F}_{t-1}$-measurable way. As the procedure stops after finite steps one can assume that $(\theta_n(1))$ is bounded which proves the lemma.

$\square$

In the next lemma we shall us the no-arbitrage condition:

Lemma 5 *If L is the subspace of the previous lemma and if*

$$L \cap L^0_+(\Omega, \mathcal{A}, \mathbf{P}) = \{0\},$$

then the cone

$$C \mathrel{\hat{=}} L - L^0_+(\Omega, \mathcal{A}, \mathbf{P})$$

is closed in $L^0(\Omega, \mathcal{A}, \mathbf{P})$.

Proof: One can proof the lemma with a modification of the proof of the previous lemma: One proves the lemma again with induction. Let us first assume that $T = 1$ and let

$$a_n \mathrel{\hat{=}} l_n - r_n, \quad l_n \in L, r_n \geq 0$$

be a convergent sequence in C. Again the problem is that the convergence of (a_n) does not imply the convergence of $l_n \mathrel{\hat{=}} (X, y_n)$. Again the problem is that the sequence (y_n) is not necessarily bounded. After dividing by $\|y_n\|$ and selecting a subsequence we can choose (y_n) in such a way that (l_n) is convergent.

One can observe that this implies that the sequence $(r_{\sigma_n}/\|y_{\sigma_n}\|)$ is convergent and we get the relation

$$0 = (X, y_\infty) - r_\infty, \quad r_\infty \geq 0.$$

Now using the no-arbitrage condition one can easily prove that $r_\infty = 0$. From this point one should eliminate one coordinate as above and finish the proof of the lemma in the same way as we did above. Then let $T > 1$. First we remark that if the no-arbitrage condition holds for some time-horizon then it holds on every subset of the time-horizon. Again using the same argument as above one gets the equation

$$X(1)\theta^*(1) + \sum_{t=2}^{T} X(t)\theta^*(t) - r^* = 0.$$

Again by the no-arbitrage condition $r^* = 0$. If on a set $H \in \mathcal{F}_1$ with positive probability $X(1)\theta_1^* > 0$ then

$$X(1)\theta_1^*\chi_H = \sum_{t=2}^{T} X(t)(-\theta^*(t))\chi_H$$

which is an arbitrage strategy on the time interval $2 \leq t \leq T$, which is impossible. If $X(1)\theta_1^* < 0$ with positive probability on some $H \in \mathcal{F}_1$ then

$$\sum_{t=2}^{T} X(t)\theta_t^*\chi_H$$

is an arbitrage strategy. Hence almost surely

$$X(1)\theta_1^* = 0.$$

As $\theta_1^* \neq 0$ we can reduce the effective coordinates in the first time period and finish the proof in the usual way.

□

Example 6 *Without the no-arbitrage assumption the lemma is not true.*

Let $T \circeq 1$, $\Omega \circeq [0,1]$ and let $\mathcal{F}_0$ be the trivial σ-algebra and let $\mathcal{F}_1$ be the σ-algebra of the Borel measurable sets. Let $X(1,\omega) \circeq \omega$. Let

$$f_n(\omega) \circeq \begin{cases} n\omega & \text{if} \quad \omega \leq 1/n \\ 1 & \text{if} \quad \omega \geq 1/n \end{cases}.$$

Obviously

$$f_n(\omega) \leq nX(1) = n\omega \in K,$$

that is $f_n \in C$. Obviously $f_n \to 1$. We show that $f \notin C$. As $\mathcal{F}_0$ is the trivial σ-algebra the $\mathcal{F}_0$-measurable variables are constant, so for every $f \in C$ there is an m, such that for every ω

$$f \leq mX(1) = m\omega,$$

which is not true for $f = 1$.

□

1.3 The Kreps–Yan separation theorem

The proof of the theorem is based on some infinite dimensional separation theorem. In finite dimension, [12] it is sufficient to separate the set

$$K \circeq \left\{ H : H = \sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t) \right) \right\}$$

from the convex cone $\mathbb{R}_{+}^{m}$. In the general case one can separate two disjoint convex sets with a continuous linear functional only when one of the convex sets has an interior point. In the space L^1 the cone of the non-negative variables does not have an interior point. The next lemma solves this problem[6]:

Lemma 7 (Kreps-Yan) *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be an arbitrary probability space. Let C be a closed convex cone in $L^1(\Omega, \mathcal{A}, \mathbf{P})$. Assume that $C \supseteq \left(-L_{+}^{1}\right)$ and that*

$$C \cap L_{+}^{1} = \{0\}.$$

Then there is a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{A})$ which is equivalent[7] to the original probability measure $\mathbf{P}$ and

$$\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^{\infty}(\Omega, \mathcal{A}, \mathbf{P}),$$

and

$$\mathbf{E}^{\mathbf{Q}}(c) \circeq \int_{\Omega} c \, d\mathbf{Q} = \int_{\Omega} c \frac{d\mathbf{Q}}{d\mathbf{P}} d\mathbf{P} = \mathbf{E}^{\mathbf{P}}\left(c \cdot \frac{d\mathbf{Q}}{d\mathbf{P}} \right) \leq 0, \quad \forall c \in C.$$

Proof: The dual of L^1 is L^{∞}, hence every continuous linear functional defined on L^1 has a representation as an integral with some function $z \in L^{\infty}$:

$$\langle z, l \rangle = \int_{\Omega} z \cdot l \, d\mathbf{P}, \quad \forall l \in L^1.$$

Let $\mathcal{Z}$ be the set of functions $z \in L^{\infty}$ with the property that $\langle z, C \rangle \leq 0$. As $0 \in \mathcal{Z}$ trivially $\mathcal{Z} \neq \emptyset$. By the assumptions of the lemma $C \supseteq \left(-L_{+}^{1}\right)$, therefore if $z \in \mathcal{Z}$ then $z \geq 0$: If on a set B with positive measure $z_x < 0$ then

$$\langle z_x, -\chi_B \rangle = -\int_B z_x d\mathbf{P} > 0,$$

which is impossible as $-\chi_B \in -L_{+}^{1} \subseteq C$.

Let $\mathcal{Y}$ denotes the set of positivity of the functions from $\mathcal{Z}$, that is $Y \in \mathcal{Y}$, if and only if there is a function $z \in \mathcal{Z}$, such that $Y = \{z > 0\}$. Obviously $\mathcal{Y}$ is

[6] See: [11], [13], [17].

[7] Recall that $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if $\mathbf{P}(A) = 0$ exactly when $\mathbf{Q}(A) = 0$. This is equivalent to the condition that $d\mathbf{Q}/d\mathbf{P}$ exists and it is positive.

closed for the countable union: If $z_n \in \mathcal{Z}$ then there are positive constants α_n with $\sum_n \alpha_n z_n \in \mathcal{Z}$. If

$$\lambda_0 = \sup \{\mathbf{P}(Y) : Y \in \mathcal{Y}\},$$

then there is a sequence (Y_n) with $\mathbf{P}(Y_n) \nearrow \lambda_0$. Without loss of generality one can assume that (Y_n) is increasing. As $\mathcal{Y}$ is closed for the countable union $Y_0 \doteq \cup_n Y_n \in \mathcal{Y}$. Obviously $\mathbf{P}(Y_0) = \lambda_0$. To proof the lemma one should show that $\lambda_0 = 1$. (As in this case there is a function $z_0 \in \mathcal{Z} \subseteq L^\infty$ for which $\langle z_0, C\rangle \leq 0$ and $\mathbf{P}(z_0 > 0) = 1$, therefore if $d\mathbf{Q}/d\mathbf{P} \doteq z_0$ then $\mathbf{Q}$ satisfies the stated properties.)

Assume that $\mathbf{P}(Y_0) < 1$ and let $x \doteq \chi_{Y_0^c} \in L^1_+ \setminus \{0\}$. The condition $C \cap L^1_+ = \{0\}$ trivially implies that $x \notin C$. As by the assumptions of the lemma C is closed and convex by the Hahn–Banach theorem there is a continuos linear functional z_x over L^1 which strictly separates x and C :

$$\langle z_x, x\rangle > \langle z_x, c\rangle, \quad \forall c \in C. \tag{2}$$

C is a cone, hence if $\langle z_x, c\rangle > 0$ for some $c \in C$ then $\langle z_x, \lambda \cdot c\rangle \nearrow \infty$ if $\lambda \nearrow \infty$ which is impossible by (2). Hence

$$\langle z_x, c\rangle \leq 0, \quad \forall c \in C$$

that is $z_x \in \mathcal{Z}$. On the other hand $0 \in C$, therefore $\langle z_x, x\rangle > 0$, that is

$$\langle z_x, x\rangle = \int_\Omega z_x \cdot x \, d\mathbf{P} = \int_\Omega z_x \cdot \chi(Y_0^c) \, d\mathbf{P} = \int_{Y_0^c} z_x \, d\mathbf{P} > 0. \tag{3}$$

This means that with positive probability z_x is positive on a subset of Y_0^c. Obviously $z_0 + z_x \in \mathcal{Z}$ as

$$\langle z_0 + z_x, C\rangle = \langle z_0, C\rangle + \langle z_x, C\rangle \leq 0$$

and by (3) the set of positivity of $z_0 + z_x$ is larger than Y_0 which contradicts to the maximality of $\mathbf{P}(Y_0)$.

□

1.4 The proof of the theorem

Finally we give the proof of the theorem[8].

1. If the no-arbitrage condition holds then C is closed by Lemma 5.

2. From the second statement the third statement is trivial.

3. Observe that for every random variable η one can change the probability measure $\mathbf{P}$ in such a way that η will be integrable under the new probability measure: It is sufficient to define[9]

$$\mathbf{P}'(A) \doteq c \int_A \exp(-\|\eta\|) \, d\mathbf{P}, \quad A \in \mathcal{A}$$

[8] See.: [10].

[9] The function $x \exp(-|x|)$ is bounded and the Radon–Nikodym derivative is bounded.

which is equivalent to $\mathbf{P}$. As the conditions of the theorem do not change if we change the probability measure in an equivalent way[10], without loss of generality one can assume that S is integrable in every period of time. As the convergence in L^1 implies the convergence in probability the cone $C_0 \circeq \mathrm{cl}(C) \cap L^1$ is closed in L^1. By the no-arbitrage condition $C_0 \cap L^1_+ = \{0\}$. By the Kreps–Yan theorem there is an equivalent probability measure $\mathbf{Q}$ with $d\mathbf{Q}/d\mathbf{P} \in L^\infty$ for which

$$\mathbf{E}^{\mathbf{Q}}(k) \leq 0, \quad \forall k \in C_0.$$

Using this for $k \circeq \pm (S(t) - S(t-1), \theta(t))$ where $\theta(t)$ is $\mathcal{F}_{t-1}$-measurable one gets that

$$\mathbf{E}^{\mathbf{Q}}((S(t) - S(t-1), \theta(t))) = 0.$$

If $\theta(t) \circeq \chi_F$ where $F \in \mathcal{F}_{t-1}$ then by the definition of the conditional expectation

$$\mathbf{E}^{\mathbf{Q}}(S(t) - S(t-1) \mid \mathcal{F}_{t-1}) = 0,$$

that is S is a martingale under $\mathbf{Q}$.

4. Finally let us assume that there is an equivalent measure $\mathbf{Q}$ and S is a martingale under $\mathbf{Q}$. If $h \in C \cap L^0_+$ then there is a predictable process θ such that

$$0 \leq h \leq \sum_{t=1}^{T} (S(t) - S(t-1), \theta(t)). \tag{4}$$

It is sufficient to show that

$$0 \leq \mathbf{E}^{\mathbf{Q}}(h) \leq \mathbf{E}^{\mathbf{Q}}\left(\sum_{t=1}^{T} (S(t) - S(t-1), \theta(t))\right) = 0 \tag{5}$$

which implies that h is almost surely zero under $\mathbf{Q}$. But $\mathbf{P}$ and $\mathbf{Q}$ are equivalent hence h is almost surely zero under $\mathbf{P}$. Hence the first statement holds.

The proof of (5) is not entirely trivial as the strategies $\theta(t)$ are not necessarily bounded, so one cannot take out the measurable components from the conditional expectation and we cannot even assume that the expressions

$$(S(t) - S(t-1), \theta(t))$$

are integrable. On the other hand with a simple argument we can solve this problem[11]. Let $\varepsilon > 0$ be arbitrary. Let us multiply the line (4) with $\chi(\|\theta(1)\| \leq n_1)$. Obviously $\theta(1)\chi(\|\theta(1)\| \leq n_1)$ is bounded, $\chi(\|\theta(1)\| \leq n_1)$ is $\mathcal{F}_0$-measurable for every n, so the strategy $\theta\chi(\|\theta(1)\| \leq n_1)$ is predictable. As S is a martingale

[10] A sequence is convergent in probability if and only if its every subsequence has a subsequence which is convergent to the same variable almost surely. Therefore an equivalent change of measure is not changing the convergent sequences in L^0.

[11] See: [5], [6]. One can also use Proposition 15, page 24.

under $\mathbf{Q}$

$$\mathbf{E}^{\mathbf{Q}}\left(\left(S\left(1\right)-S\left(0\right),\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(1\right)\right)\right)=$$
$$\mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(\left(S\left(1\right)-S\left(0\right),\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(1\right)\right)\mid\mathcal{F}_{0}\right)\right)=$$
$$\mathbf{E}^{\mathbf{Q}}\left(\left(\mathbf{E}^{\mathbf{Q}}\left(S\left(1\right)-S\left(0\right)\mid\mathcal{F}_{0}\right),\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(1\right)\right)\right)=$$
$$\mathbf{E}^{\mathbf{Q}}\left(\left(0,\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(1\right)\right)\right)=0.$$

Now using the additivity for integrable variables

$$0\leq\mathbf{E}^{\mathbf{Q}}\left(h\chi\left(\left\|\theta\left(1\right)\right\|\leq n\right)\right)\leq$$
$$\leq\mathbf{E}^{\mathbf{Q}}\left(\left(S\left(1\right)-S\left(0\right),\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(1\right)\right)\right)+$$
$$+\mathbf{E}^{\mathbf{Q}}\left(\sum_{t=2}^{T}\left(S\left(t\right)-S\left(t-1\right),\chi\left(\left\|\theta\left(1\right)\right\|\leq n_{1}\right)\theta\left(t\right)\right)\right)$$

where the first expected value is zero. Now let us multiply (4) by $\chi\left(\left\|\theta\left(2\right)\right\|\leq n_{2}\right)$. Using the Dominated Convergence Theorem for some n_{2}

$$\mathbf{E}^{\mathbf{Q}}\left(h\prod_{t=1}^{2}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\leq$$
$$\leq\mathbf{E}^{\mathbf{Q}}\left(\left(S\left(1\right)-S\left(0\right),\theta\left(1\right)\prod_{t=1}^{2}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\right)+$$
$$+\mathbf{E}^{\mathbf{Q}}\left(\sum_{t=2}^{T}\left(S\left(t\right)-S\left(t-1\right),\theta\left(t\right)\prod_{t=1}^{2}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\right)<$$
$$<\frac{\varepsilon}{T}+\mathbf{E}^{\mathbf{Q}}\left(\sum_{t=2}^{T}\left(S\left(t\right)-S\left(t-1\right),\theta\left(t\right)\prod_{t=1}^{2}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\right)=$$
$$=\frac{\varepsilon}{T}+\mathbf{E}^{\mathbf{Q}}\left(\sum_{t=3}^{T}\left(S\left(t\right)-S\left(t-1\right),\theta\left(t\right)\prod_{t=1}^{2}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\right).$$

Continuing the procedure one can show that

$$\mathbf{E}^{\mathbf{Q}}\left(h\prod_{t=1}^{T}\chi\left(\left\|\theta\left(t\right)\right\|\leq n_{t}\right)\right)\leq\varepsilon.$$

By the Monotone Convergence Theorem

$$\mathbf{E}^{\mathbf{Q}}\left(h\right)\leq\varepsilon,$$

hence $\mathbf{E}^{\mathbf{Q}}\left(h\right)=0$.

□

Finally let us make a comment:

One can prove the existence of the martingale measure without the proof of the closedness of set C for $T > 1$. To prove the theorem for $T = 1$ it is sufficient to use Stricker's lemma. By the induction hypothesis let assume that there are a martingale measures for the time-horizons $t = 0, 1$ and $t = 1, \ldots, T$. Let $\mathbf{Q}_1$ be the martingale measure for the time-horizon $t = 0, 1$ and $\mathbf{Q}_2$ be the martingale measure for the time-horizon $t = 1, \ldots, T$. As we can apply the Kreps–Yan theorem on the space $(\Omega, \mathcal{F}_1, \mathbf{Q}_2)$ the derivative $d\mathbf{Q}_1/d\mathbf{Q}_2$ is bounded. Let

$$\frac{d\mathbf{Q}}{d\mathbf{P}} \circeq \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2}\frac{d\mathbf{Q}_2}{d\mathbf{P}}.$$

This means that

$$\mathbf{Q}(A) = \int_A \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2}\frac{d\mathbf{Q}_2}{d\mathbf{P}} d\mathbf{P} = \int_A \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2.$$

As $d\mathbf{Q}/d\mathbf{P}$ is obviously positive and bounded it is sufficient to show that $\mathbf{Q}$ is a martingale measure for the whole time-horizon $t = 0, 1, \ldots, T$. First we show that the variables $S(t)$ are integrable with respect to $\mathbf{Q}$. If $t = 0$ then

$$\begin{aligned}
\int_\Omega S(0)\, d\mathbf{Q} &\circeq \int_\Omega S(0) \frac{d\mathbf{Q}}{d\mathbf{P}} d\mathbf{P} \circeq \\
&\circeq \int_\Omega S(0) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2}\frac{d\mathbf{Q}_2}{d\mathbf{P}} d\mathbf{P} = \\
&= \int_\Omega \left(S(0) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} \right) \frac{d\mathbf{Q}_2}{d\mathbf{P}} d\mathbf{P} = \\
&= \int_\Omega S(0) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2 = \\
&= \int_\Omega S(0)\, d\mathbf{Q}_1 < \infty,
\end{aligned}$$

as $\mathbf{Q}_1$ is a martingale measure for $S(0), S(1)$. One can use the same argument if $t = 1$. If $t > 1$ then

$$\int_\Omega S(t)\, d\mathbf{Q} = \int_\Omega S(0) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2 < \infty$$

as $S(t)$ is integrable with respect to $\mathbf{Q}_2$ and the derivative $d\mathbf{Q}_1/d\mathbf{Q}_2$ is bounded. Let now $F \in \mathcal{F}_0$. Calculating as above using that $\mathbf{Q}_1$ is a martingale measure in the time-horizon $t = 0, 1$

$$\begin{aligned}
\int_F S(1)\, d\mathbf{Q} &= \int_F S(1) \frac{d\mathbf{Q}}{d\mathbf{P}} d\mathbf{P} \circeq \int_F \left(S(1) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} \right) \frac{d\mathbf{Q}_2}{d\mathbf{P}} d\mathbf{P} = \\
&= \int_F S(1) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2 = \int_F S(1)\, d\mathbf{Q}_1 = \int_F S(0)\, d\mathbf{Q}_1 = \\
&= \int_F S(0) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2 = \int_F S(0) \frac{d\mathbf{Q}}{d\mathbf{P}} d\mathbf{P} = \\
&= \int_F S(0)\, d\mathbf{Q},
\end{aligned}$$

hence

$$\mathbf{E}^{\mathbf{Q}}\left(S\left(1\right) \mid \mathcal{F}_0\right) = S\left(0\right).$$

Now let $t \geq 1$. As $d\mathbf{Q}_1/d\mathbf{Q}_2$ is bounded and it is $\mathcal{F}_1$-measurable, therefore it is $\mathcal{F}_t$-measurable as $t \geq 1$. Hence if $F \in \mathcal{F}_t$ then

$$\begin{aligned}
\int_F S\left(t+1\right) d\mathbf{Q} &= \int_F S\left(t+1\right) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} d\mathbf{Q}_2 = \\
&= \int_F \mathbf{E}^{\mathbf{Q}_2}\left(S\left(t+1\right) \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} \mid \mathcal{F}_t\right) d\mathbf{Q}_2 = \\
&= \int_F \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} \mathbf{E}^{\mathbf{Q}_2}\left(S\left(t+1\right) \mid \mathcal{F}_t\right) d\mathbf{Q}_2 = \\
&= \int_F \frac{d\mathbf{Q}_1}{d\mathbf{Q}_2} S\left(t\right) d\mathbf{Q}_2 = \int_F S\left(t\right) d\mathbf{Q},
\end{aligned}$$

which means that $\left(S\left(t\right)\right)_{t=0}^T$ is a martingale under $\mathbf{Q}$.

2 The completeness of the market and the second fundamental theorem of asset pricing

The completeness of the market is a very important concept of asset pricing. The completeness of a market means that one can hedge every future claim.

Definition 8 *The market defined by the processes S over the time-horizon $t = 0, 1, 2, \ldots, T$ is complete, if for every $\mathcal{F}_T$-measurable claim H_T there is a predictable strategy*

$$\left(\theta_i\left(t\right)\right)_{i=1}^m, \quad t = 1, \ldots, T$$

and a real number λ such that

$$H_T = \lambda + \sum_{t=1}^{T} \left(S\left(t\right) - S\left(t-1\right), \theta\left(t\right)\right).$$

Theorem 9 (Second fundamental theorem of asset pricing) *Assume that there is no arbitrage on the market defined by*

$$S = \left(S_i\left(t\right)\right)_{i=1}^m, \quad t = 0, \ldots, T.$$

The market represented by S is complete if and only if the martingale measure of S is unique on the space $\left(\Omega, \mathcal{F}_T\right)$.

Proof: The proof contains two steps.

1. Assume that the market is complete and $\mathbf{Q}$ and $\mathbf{R}$ are two different martingale measures. As the two measures are different there is a set $F \in \mathcal{F}_T$, with $\mathbf{Q}\left(F\right) \neq$

$\mathbf{R}(F)$. As the market is complete there is an m-dimensional predictable strategy $(\theta(t))_{t=1}^{T}$ such that

$$\chi_F = \lambda + \sum_{t=1}^{T} (S(t) - S(t-1), \theta(t)). \tag{6}$$

The basic idea of the proof is that we calculate the expected value on both sides with respect to measures $\mathbf{Q}$ and $\mathbf{R}$. The key observation is that for every martingale measure $\mathbf{P}$

$$\mathbf{E}^{\mathbf{P}}\left(\sum_{t=1}^{T} (S(t) - S(t-1), \theta(t))\right) = 0, \tag{7}$$

which implies that

$$\mathbf{Q}(F) = \lambda = \mathbf{R}(F),$$

which is impossible. As in the proof of the first fundamental theorem one cannot assume in (7) that the strategies θ are bounded hence one cannot necessarily use the additivity of the integral. As one can recall, the main problem with stochastic integrals is that they are not necessarily martingales but only local martingales. Hence the sum in (6) is not a martingale but just a local martingale. One can of course fix this problem in a similar way as we did during the proof of the first fundamental theorem, but we want to show the "bigger picture". As we shall show in the next subsection[12], on discrete and finite time-horizon if the values of a local martingale are integrable then the local martingale is a martingale. As χ_F is integrable the "stochastic integral" in (7) is a martingale so the expected value is zero.

2. Now assume that the market is not complete. As we assumed on the market there is no-arbitrage, so we have such a measure $\mathbf{Q}$ that the coordinates of S are martingale under $\mathbf{Q}$. By definition let

$$L \triangleq \left\{\lambda + \sum_{t=1}^{T} (S(t) - S(t-1), \theta(t))\right\},$$

where θ is predictable and λ is a real number. As the market is not complete $L \neq L^0(\Omega, \mathcal{F}_T, \mathbf{Q})$. Let H_T be a claim which is not in L. As one can always change the measure in such a way that every fixed finite number of variables become integrable under the new measure one can assume that all the variables in the model, S and H_T are integrable. By the first fundamental theorem one can assume that the Radon–Nikodym derivative $d\mathbf{Q}/d\mathbf{P}$ is bounded so one can assume that the columns of $(S(t))_{t=1}^{T}$ and H_T are integrable under the martingale measure $\mathbf{Q}$.

We prove that L is closed in $L^1(\Omega, \mathcal{F}_T, \mathbf{Q})$. Recall that the linear subspace

$$K \triangleq \left\{\sum_{t=1}^{T} (S(t) - S(t-1), \theta(t))\right\}$$

[12]See: Proposition 15, page 24.

is a closed subset of L^0. As by Markov's inequality convergence in L^1 implies the stochastic convergence $K \cap L^1$ is closed subspace of L^1. As the measures are probability measure $1 \in L^1$, so if L also denotes the intersection of L and L^1 then L is a direct sum of a closed subspace K and of a one-dimensional subspace. If $1 \in K$ then L is closed. If $1 \notin K$ then every $l \in L$ has a representation $l = \lambda 1 + r$. If $l_n \to l_\infty$ in L then the only problem is that we do not know that the sequence of scalars (λ_n) is bounded. Let d be the distance of K and of 1. As K is closed and as $1 \notin K$ obviously $d > 0$. The sequence (l_n) is convergent, so it is bounded. Assume that $\|l_n\|_1 \leq c$ for every n. As K is a subspace if $r_n \in K$ then

$$\left(-\frac{r_n}{\lambda_n}\right) \in K$$

so

$$c \geq |\lambda_n 1 + r_n| = |\lambda_n| \left|1 + \frac{r_n}{\lambda_n}\right| = |\lambda_n| \left|1 - \left(-\frac{r_n}{\lambda_n}\right)\right| \geq |\lambda_n| \, d.$$

But $d > 0$, so

$$\frac{c}{d} \geq |\lambda_n| \, ,$$

which implies that (λ_n) is bounded. Hence sequence (λ_n) has a convergent subsequence. Denoting this subsequence also with (λ_n) one has that $(\lambda_n 1)$ is also convergent. As the sum $l_n = \lambda_n 1 + r_n$ is a convergent sequence (r_n) is also convergent. As K is closed the limit of (r_n) is in K. So the limit of a subsequence $(\lambda_n 1 + r_n)$ is in L, therefore the limit of $(\lambda_n 1 + r_n)$ is in L. Hence L is closed.

As $H_T \notin L$ is integrable, there is a function in L^1 which is not in the closed subspace L. By the Hahn–Banach theorem there is a $z \in L^\infty(\Omega, \mathcal{F}_T, \mathbf{Q})$, which separates the subspace L and the variable H_T. As L is a subspace

$$\langle z, l \rangle \doteq \int_\Omega z \cdot l d\mathbf{Q} = \mathbf{E}^{\mathbf{Q}}(z \cdot l) = 0, \quad l \in L. \tag{8}$$

As $\theta(t) = 0$ and $\lambda = 1$ is a predictable strategy $1 \in L$

$$\langle z, 1 \rangle \doteq \int_\Omega z \cdot 1 d\mathbf{Q} = \int_\Omega z d\mathbf{Q} = 0.$$

Let

$$g \doteq 1 + \frac{z}{2\|z\|_\infty} \geq \frac{1}{2} > 0,$$

and let

$$\mathbf{R}(A) \doteq \int_A g \, d\mathbf{Q}.$$

The derivative $g \doteq d\mathbf{R}/d\mathbf{Q}$ is bounded from above and it is bounded from below with a positive number, so the integrable variable under $\mathbf{R}$ and $\mathbf{Q}$ are the same.

$$\mathbf{R}(\Omega) = \mathbf{E}^{\mathbf{Q}}(1) + \frac{\mathbf{E}^{\mathbf{Q}}(z)}{2\|z\|_\infty} = 1,$$

so $\mathbf{R}$ is an equivalent probability measure. For an arbitrary predictable strategy θ and for the constant $\lambda = 0$

$$\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t)\right) \in L,$$

hence if θ is bounded then using (8)

$$\begin{aligned} & \mathbf{E}^{\mathbf{R}} \left(\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t)\right) \right) \mathrel{\mathring{=}} \\ \mathrel{\mathring{=}} \; & \mathbf{E}^{\mathbf{Q}} \left(\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t)\right) \left(1 + \frac{z}{2\left\|z\right\|_{\infty}}\right) \right) = \\ = \; & \mathbf{E}^{\mathbf{Q}} \left(\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t)\right) \right). \end{aligned}$$

As S is a martingale under $\mathbf{Q}$ and as θ is predictable the expression on the right-hand side is

$$\begin{aligned} & \mathbf{E}^{\mathbf{Q}} \left(\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta(t)\right) \right) = \\ & = \sum_{t=1}^{T} \mathbf{E}^{\mathbf{Q}} \left(\left(S(t) - S(t-1), \theta(t)\right) \right) = \\ & = \sum_{t=1}^{T} \mathbf{E}^{\mathbf{Q}} \left(\mathbf{E}^{\mathbf{Q}} \left(\left(S(t) - S(t-1), \theta(t)\right) \mid \mathcal{F}_{t-1} \right) \right) = \\ & = \sum_{t=1}^{T} \mathbf{E}^{\mathbf{Q}} \left(\left(\mathbf{E}^{\mathbf{Q}} \left(S(t) - S(t-1) \mid \mathcal{F}_{t-1}\right), \theta(t) \right) \right) = \\ & = \sum_{t=1}^{T} \mathbf{E}^{\mathbf{Q}} \left(\left(0, \theta(t)\right) \right) = 0. \end{aligned}$$

for every bounded predictable θ. So the left-hand side is also zero. If θ is zero outside time period $t-1$ where it is χ_F, where $F \in \mathcal{F}_{t-1}$ then

$$\mathbf{E}^{\mathbf{R}} \left(\left(S(t) - S(t-1)\right) \chi_F \right) = 0,$$

which is the same as

$$\int_F S(t)\, d\mathbf{R} = \int_F S(t-1)\, d\mathbf{R}.$$

Hence by the definition of the conditional expectation

$$\mathbf{E}^{\mathbf{R}} \left(S(t) \mid \mathcal{F}_{t-1}\right) = S(t-1).$$

Hence S is a martingale under the measure $\mathbf{R} \neq \mathbf{Q}$ so the martingale measure is not unique.

□

2.1 Discrete-time local martingales

In this subsection we shortly recall some properties of discrete-time local martingales[13].

If ξ is a non-negative random variable and $\mathcal{F}$ is a σ-algebra then the conditional expectation $\mathbf{E}(\xi \mid \mathcal{F})$ is always well-defined. In this case one can easily prove that the conditional expectation is, monotone, additive and one can use the tower law. One can also show that if ξ and η are non-negative and if ξ is $\mathcal{F}$-measurable then

$$\mathbf{E}(\xi\eta \mid \mathcal{F}) = \xi\mathbf{E}(\eta \mid \mathcal{F}). \tag{9}$$

If ξ does not have a finite expected value then it can happen that the conditional expectation $\mathbf{E}(\xi \mid \mathcal{F})$ is not a random variable that is it is not almost surely finite. This explains the next definition:

Definition 10 *Let ξ be a random variable and let $\mathcal{F}$ be a σ-algebra. If ξ^{+} and ξ^{-} both have finite-valued conditional expectation with respect to $\mathcal{F}$ then the expression*

$$\mathbf{E}\left(\xi^{+} \mid \mathcal{F}\right) - \mathbf{E}\left(\xi^{-} \mid \mathcal{F}\right)$$

is well-defined and we shall call it the generalized conditional expectation of ξ with respect to the conditional σ-algebra $\mathcal{F}$. We shall denote the generalized conditional expectation with same symbol as the ordinary conditional expectation: $\mathbf{E}(\xi \mid \mathcal{F})$.

One can easily show that the usual rules of calculation remain true for the generalized conditional expectation.

Definition 11 *We shall call the finite or infinite sequence $(\xi_n, \mathcal{F}_n)$ a discrete-time generalized martingale if*[14]

1. *the generalized conditional expectation $\mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right)$ exists for every n,*
2. *and also for every n*

$$\mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right) \circeq \mathbf{E}\left(\xi_{n+1}^{+} \mid \mathcal{F}_n\right) - \mathbf{E}\left(\xi_{n+1}^{-} \mid \mathcal{F}_n\right) = \xi_n,$$

One should emphasize that we are not assuming that the variables ξ_n have finite expected value. We are not even assuming that they have an infinite expected value. What is the difference between martingales an generalized martingales?

Definition 12 *We shall call the finite or infinite sequence $(\xi_n, \mathcal{F}_n)$ a local martingale, if there is a sequence of stopping times (τ_k) such that $\tau_k \nearrow \infty$ and the*

[13] See: [9], [15]

[14] This is the same as $\mathbf{E}\left(\xi_{n+1}^{\pm} \mid \mathcal{F}_n\right) = \xi_n^{\pm}$ for every n.

stopped processes[15]

$$\xi_n^{\tau_k} \circeq \chi(\tau_k > 0)\, \xi_{n\wedge\tau_k}$$

are martingales with respect to the filtration $(\mathcal{F}_n)$ *for every* k.

Definition 13 *We shall call the finite or infinite sequence* $(\xi_n, \mathcal{F}_n)$ *a martingale transform*[16] *if there is a martingale*

$$(M_n, \mathcal{F}_n)$$

and a predictable sequence (θ_n) *such that*[17]

$$\xi_n = \xi_0 + \sum_{k=1}^{n} \theta_k (M_k - M_{k-1}). \tag{10}$$

The structure of discrete-time local martingales is very simple[18]:

Proposition 14 *The next statements are equivalent:*

1. (ξ_n) *is a local martingale,*
2. (ξ_n) *is a generalized martingale,*
3. (ξ_n) *is a martingale transform.*

Proof: We show that implications 1. ⇒ 2. ⇒ 3. ⇒ 1. hold.

1. Let (ξ_n) be a local martingale and let (τ_k) be a localizing sequence. By the definition of the martingales the expected value of $\xi_{n+1}^{\tau_k}$ is finite, hence the conditional expectations below are also finite, so

$$\begin{aligned}
\infty &> \mathbf{E}\left(\left|\xi_{n+1}^{\tau_k}\right| \mid \mathcal{F}_n\right) \circeq \mathbf{E}\left(\left|\xi_{(n+1)\wedge\tau_k}\right| \chi(\tau_k > 0) \mid \mathcal{F}_n\right) \geq \\
&\geq \mathbf{E}\left(\left|\xi_{(n+1)\wedge\tau_k}\right| \chi(\tau_k > n) \mid \mathcal{F}_n\right) = \\
&= \mathbf{E}\left(\left|\xi_{n+1}\right| \chi(\tau_k > n) \mid \mathcal{F}_n\right) = \\
&= \chi(\tau_k > n)\, \mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right).
\end{aligned}$$

In the last line we used that as τ_k is a stopping time therefore $\chi(\tau_k > n)$ is $\mathcal{F}_n$-measurable for every n, so one can apply (9) with non-negative variables.

[15] Of course $\xi_{0\wedge\tau} = \xi_0$ for every stopping time τ. The role of $\chi(\tau_k > 0)$ is to manipulate the value of the stopped process at $n = 0$. If $\sigma_k(\omega) \circeq \infty$ when $|\xi_0| \leq k$ and zero otherwise then σ_k is a stopping time, and $\left|\xi_0^{\sigma_k}\right| \leq k$. It is easy to see that (ξ_n) is a local martingale if and only if it has the representation $\xi_n = \xi_0 + \eta_n$, where ξ_0 is an arbitrary $\mathcal{F}_0$-measurable variable and for (η_n) there is a localizing sequence $\tau_k \nearrow \infty$ such that $\left(\eta_{n\wedge\tau_k}\right)$ is a martingale for every k.

[16] The martingale transforms are the discrete-time stochastic integrals.

[17] Obviously by definition $\mathcal{F}_{-1} \circeq \mathcal{F}_0$.

[18] The proposition in fact says that on discrete-time horizon the local martingales are "badly integrable" martingales. This is not true if the time-horizon is continuous.

By the definition of the localizing sequence $\tau_k(\omega) \nearrow \infty$ for almost all ω. So for almost all ω if k is sufficiently large

$$\mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right)(\omega) = \chi\left(\tau_k(\omega) > n\right)\mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right)(\omega) < \infty.$$

Therefore $\mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) < \infty$ almost surely. Obviously $\xi_{n+1}^{\pm} \leq \left|\xi_{n+1}\right|$ so the conditional expectations $\mathbf{E}\left(\xi_{n+1}^{\pm} \mid \mathcal{F}_n\right)$ exist and they are finite. Hence by the definition of the generalized conditional expectation $\mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right)$ exists. Of course for a generalized conditional expectation the integral equations defining the classical conditional expectation are not necessarily meaningful. Let

$$\mathcal{G}_n \circeq \left\{F \in \mathcal{F}_n : \int_F \left|\xi_{n+1}\right| d\mathbf{P} < \infty\right\}.$$

It is clear from the definition of the conditional expectation that for every $F \in \mathcal{G}_n$

$$\int_F \left|\xi_{n+1}\right| d\mathbf{P} = \int_F \mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) d\mathbf{P} < \infty.$$

It is also clear from the definition that if $H \subseteq F \in \mathcal{G}_n$ then $H \in \mathcal{G}_n$ as well. By definition $\xi_n^{\tau_k}$ is a martingale, therefore $|\xi_n^{\tau_k}|$ is a submartingale, so

$$\begin{aligned}\int_{F\cap\{\tau_k>n\}} \left|\xi_n\right| d\mathbf{P} &= \int_{F\cap\{\tau_k>n\}} \left|\xi_n^{\tau_k}\right| d\mathbf{P} \leq \int_{F\cap\{\tau_k>n\}} \left|\xi_{n+1}^{\tau_k}\right| d\mathbf{P} = \\ &= \int_{F\cap\{\tau_k>n\}} \left|\xi_{n+1}\right| d\mathbf{P}.\end{aligned}$$

Hence if $k \to \infty$ then by the Monotone Convergence Theorem if $F \in \mathcal{G}_n$

$$\int_F \left|\xi_n\right| d\mathbf{P} \leq \int_F \left|\xi_{n+1}\right| d\mathbf{P} < \infty, \tag{11}$$

therefore ξ_n is also integrable over F.

$$\begin{aligned}\int_{F\cap\{\tau_k>n\}} \xi_n d\mathbf{P} &= \int_{F\cap\{\tau_k>n\}} \xi_n^{\tau_k} d\mathbf{P} = \int_{F\cap\{\tau_k>n\}} \xi_{n+1}^{\tau_k} d\mathbf{P} = \\ &= \int_{F\cap\{\tau_k>n\}} \xi_{n+1} d\mathbf{P}.\end{aligned}$$

Using the Dominated Convergence Theorem on both sides of the equation and using that both ξ_n and ξ_{n+1} are integrable over F

$$\int_F \xi_n d\mathbf{P} = \int_F \xi_{n+1} d\mathbf{P}, \quad F \in \mathcal{G}_n.$$

As on every set in $\mathcal{G}_n$ the random variables ξ_n and ξ_{n+1} are integrable and for integrable variables the integral is additive, by the definition of the conditional expectation if $F \in \mathcal{G}_n$ then

$$\begin{aligned}
\int_F \xi_{n+1} d\mathbf{P} &= \int_F \xi_{n+1}^+ - \xi_{n+1}^- d\mathbf{P} = \int_F \xi_{n+1}^+ d\mathbf{P} - \int_F \xi_{n+1}^- d\mathbf{P} = \\
&= \int_F \mathbf{E}\left(\xi_{n+1}^+ \mid \mathcal{F}_n\right) d\mathbf{P} - \int_F \mathbf{E}\left(\xi_{n+1}^- \mid \mathcal{F}_n\right) d\mathbf{P} = \\
&= \int_F \mathbf{E}\left(\xi_{n+1}^+ \mid \mathcal{F}_n\right) - \mathbf{E}\left(\xi_{n+1}^- \mid \mathcal{F}_n\right) d\mathbf{P} \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \int_F \mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right) d\mathbf{P},
\end{aligned}$$

where in the last two lines we used that by the definition of the generalized conditional expectation and by the restriction $F \in \mathcal{G}_n$ the integrals of $\mathbf{E}\left(\xi_{n+1}^{\pm} \mid \mathcal{F}_n\right)$ are also finite hence one can use the additivity of the integral. Recall that if $H \subseteq F \in \mathcal{G}_n$ then $H \in \mathcal{G}_n$ so one can apply the above calculation for H as well, so

$$\int_H \xi_n d\mathbf{P} = \int_H \mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right) d\mathbf{P}, \quad \forall H \in \mathcal{F}_n, H \subseteq F \in \mathcal{G}_n.$$

As both sides are $\mathcal{F}_n$-measurable on the sets $F \in \mathcal{G}_n$

$$\xi_n \stackrel{m.m.}{=} \mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right). \tag{12}$$

Let $G_n \stackrel{\circ}{=} \left\{\mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) \leq n\right\} \in \mathcal{F}_n$. Obviously by the definition of the conditional expectation

$$\int_{G_n} \left|\xi_{n+1}\right| d\mathbf{P} = \int_{G_n} \mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) d\mathbf{P} \leq n$$

so $G_n \in \mathcal{G}_n$. As $\mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) < \infty$ almost surely $\cup_n G_n$ is almost surely equal to Ω. As on G_n (12) holds (12) holds outside a set of measure zero, that is

$$\xi_n \stackrel{m.m.}{=} \mathbf{E}\left(\xi_{n+1} \mid \mathcal{F}_n\right).$$

Hence (ξ_n) is a generalized martingale.

2. Now assume that (ξ_n) is a generalized martingale. To make the notation simply we shall assume that that the variables (ξ_n) are everywhere finite. Let

$$A(n,k) \stackrel{\circ}{=} \left\{k \leq \mathbf{E}\left(\left|\xi_{n+1} - \xi_n\right| \mid \mathcal{F}_n\right) < k+1\right\}.$$

As (ξ_n) is a generalized martingale

$$\mathbf{E}\left(\left|\xi_{n+1} - \xi_n\right| \mid \mathcal{F}_n\right) \leq \mathbf{E}\left(\left|\xi_{n+1}\right| \mid \mathcal{F}_n\right) + \mathbf{E}\left(\left|\xi_n\right| \mid \mathcal{F}_n\right) < \infty$$

therefore $A(n,k)$ is a partition of Ω for every n, that is the union of sets $A(n,k)$ with respect to k is Ω, and for two different k the intersection is empty. Let

$$u_n \stackrel{\circ}{=} \sum_{k \geq 0} \frac{1}{(k+1)^3} \left(\xi_n - \xi_{n-1}\right) \chi_{A(n-1,k)}.$$

As the sets $(A(n-1,k))_k$ form a partition the definition of u_n is correct. Obviously u_n is finite and it is $\mathcal{F}_n$-measurable. As $(A(n-1,k))_k$ is a partition

$$|u_n| = \sum_{k\geq 0} \frac{1}{(k+1)^3} \left|\xi_n - \xi_{n-1}\right| \chi_{A(n-1,k)}.$$

Taking conditional expectation with respect to $\mathcal{F}_{n-1}$ on both sides and using the Monotone Convergence Theorem for the conditional expectation, the additivity of the conditional expectation among non-negative variables and (9)

$$\mathbf{E}\left(|u_n| \mid \mathcal{F}_{n-1}\right) = \sum_{k\geq 0} \frac{\chi_{A(n-1,k)}}{(k+1)^3} \mathbf{E}\left(\left|\xi_n - \xi_{n-1}\right| \mid \mathcal{F}_{n-1}\right) \leq \sum_{k\geq 0} \frac{1}{(k+1)^2} < \infty.$$

Therefore, using the tower law for non-negative variables

$$\mathbf{E}\left(|u_n|\right) = \mathbf{E}\left(\mathbf{E}\left(|u_n| \mid \mathcal{F}_{n-1}\right)\right) \leq \sum_{k\geq 0} \frac{1}{(k+1)^2} < \infty. \tag{13}$$

Hence u_n is integrable. For every k

$$\left|\xi_n - \xi_{n-1}\right| \chi_{A(n-1,k)}$$

is also integrable hence using that for integrable variable the conditional expectation and generalized conditional expectation is the same

$$\begin{aligned}
&\mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right) \chi_{A(n-1,k)} \mid \mathcal{F}_{n-1}\right) = \\
&= \mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{+} \chi_{A(n-1,k)} \mid \mathcal{F}_{n-1}\right) - \mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{-} \chi_{A(n-1,k)} \mid \mathcal{F}_{n-1}\right) = \\
&= \chi_{A(n-1,k)} \mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{+} \mid \mathcal{F}_{n-1}\right) - \chi_{A(n-1,k)} \mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{-} \mid \mathcal{F}_{n-1}\right) = \\
&= \chi_{A(n-1,k)} \left(\mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{+} \mid \mathcal{F}_{n-1}\right) - \mathbf{E}\left(\left(\xi_n - \xi_{n-1}\right)^{-} \mid \mathcal{F}_{n-1}\right)\right) \circeq \\
&\circeq \chi_{A(n-1,k)} \mathbf{E}\left(\xi_n - \xi_{n-1} \mid \mathcal{F}_{n-1}\right) = \\
&= \chi_{A(n-1,k)} \left(\mathbf{E}\left(\xi_n \mid \mathcal{F}_{n-1}\right) - \mathbf{E}\left(\xi_{n-1} \mid \mathcal{F}_{n-1}\right)\right) = \\
&= \chi_{A(n-1,k)} \left(\mathbf{E}\left(\xi_n \mid \mathcal{F}_{n-1}\right) - \xi_{n-1}\right) = 0.
\end{aligned}$$

Using the Dominated Convergence Theorem for the conditional expectation and the line (13)

$$\mathbf{E}\left(u_n \mid \mathcal{F}_{n-1}\right) = \mathbf{E}\left(\sum_{k=0}^{\infty} \frac{1}{(k+1)^3} \left(\xi_n - \xi_{n-1}\right) \chi_{A(n-1,k)} \mid \mathcal{F}_{n-1}\right) = 0.$$

Hence (u_n) is a sequence of martingale differences, therefore

$$M_n \circeq \sum_{k=1}^{n} u_k$$

is a martingale. Let

$$\theta_n \mathrel{\mathring{=}} \sum_{k\geq 0} (k+1)^3 \chi_{A(n-1,k)}.$$

As the sets $A(n-1,k)$ are disjoint and they are $\mathcal{F}_{n-1}$-measurable θ_n is well-defined and it is predictable. Again using that the sets $A(n-1,k)$ are disjoint

$$(M_n - M_{n-1})\,\theta_n = u_n\theta_n =$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^3} \left(\xi_n - \xi_{n-1}\right) \chi_{A(n-1,k)} \sum_{k=0}^{\infty} (k+1)^3 \chi_{A(n-1,k)} =$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)^3}{(k+1)^3} \chi_{A(n-1,k)} \left(\xi_n - \xi_{n-1}\right) = \xi_n - \xi_{n-1}.$$

Hence

$$\sum_{k=1}^{n} (M_n - M_{n-1})\,\theta_n = \xi_n - \xi_0$$

therefore (ξ_n) is the martingale transform of (M_n) with the predictable sequence (θ_n).

3. Finally assume that (ξ_n) is a martingale transform and let assume that (10) holds. Let

$$\tau_k \mathrel{\mathring{=}} \inf\{n \geq 0 : |\theta_{n+1}| > k\}.$$

By definition

$$\begin{aligned}
\{\tau_k = 0\} &= \{|\theta_1| > k\} \\
\{\tau_k = 1\} &= \{|\theta_1| \leq k\} \cap \{|\theta_2| > k\} \\
\{\tau_k = 2\} &= \{|\theta_1| \leq k\} \cap \{|\theta_2| \leq k\} \cap \{|\theta_3| > k\} \\
&\vdots
\end{aligned}$$

which implies that τ_k is a stopping time for every k. As (θ_n) is predictable and as τ_k is a stopping time the stopped sequences $(\theta_n^{\tau_k})_n$ are also predictable: for every n and α

$$\{\theta_n^{\tau_k} < \alpha\} =$$

$$= (\{\theta_n < \alpha\} \cap \{\tau_k \geq n\}) \cup (\{\theta_1 < \alpha\} \cap \{\tau_k = 1\}) \cup \ldots$$

$$\cup (\{\theta_{n-1} < \alpha\} \cap \{\tau_k = n-1\}).$$

and as

$$\{\tau_k \geq n\} = \{\tau_k < n\}^c = \{\tau_k \leq n-1\}^c \in \mathcal{F}_{n-1},$$

one obviously gets that

$$\{\theta_n^{\tau_k} < \alpha\} \in \mathcal{F}_{n-1},$$

which means that $(\theta_n^{\tau_k})_n$ is predictable. The stopped martingales are martingales, so

$$\mathbf{E}\left((M_{n+1} - M_n)^{\tau_k} \mid \mathcal{F}_{n-1}\right) = 0.$$

$\theta_n^{\tau_k}$ is $\mathcal{F}_{n-1}$-measurable and bounded, therefore

$$\begin{aligned}\mathbf{E}\left(\xi_{n+1}^{\tau_k}-\xi_n^{\tau_k}\mid\mathcal{F}_{n-1}\right) &= \mathbf{E}\left(\left(\xi_{n+1}-\xi_n\right)^{\tau_k}\mid\mathcal{F}_{n-1}\right)= \\ &= \mathbf{E}\left(\left(\theta_n\left(M_{n+1}-M_n\right)\right)^{\tau_k}\mid\mathcal{F}_{n-1}\right) \\ &= \mathbf{E}\left(\theta_n^{\tau_k}\left(M_{n+1}-M_n\right)^{\tau_k}\mid\mathcal{F}_{n-1}\right)= \\ &= \theta_n^{\tau_k}\mathbf{E}\left(\left(M_{n+1}-M_n\right)^{\tau_k}\mid\mathcal{F}_{n-1}\right)=0.\end{aligned}$$

This means that $(\xi_n^{\tau_k})$ is a martingale, that is (ξ_n) is a local martingale.

□

Using the proposition one can prove the statement used during the proof of the second fundamental theorem:

Proposition 15 *If $(\xi_n)_{n=0}^T$ is a martingale transform and ξ_T is integrable then the sequence $(\xi_n)_{n=0}^T$ is a martingale..*

Proof: By the just proved implication 3.⇒2. in Proposition 14

$$\xi_{T-1}=\mathbf{E}\left(\xi_T\mid\mathcal{F}_{T-1}\right),$$

where of course $\mathbf{E}$ denotes the generalized conditional expectation. As ξ_T integrable, by the trivial application of the tower law

$$\mathbf{E}\left(\left|\xi_{T-1}\right|\right)=\mathbf{E}\left(\left|\mathbf{E}\left(\xi_T\mid\mathcal{F}_{T-1}\right)\right|\right)\leq\mathbf{E}\left(\mathbf{E}\left(\left|\xi_T\right|\mid\mathcal{F}_{T-1}\right)\right)=\mathbf{E}\left(\left|\xi_T\right|\right)<\infty$$

hence ξ_{T-1} is also integrable. From this point the proof of the proposition is obvious.

□

2.2 Pricing European derivatives on discrete and finite time-horizon

With the first and the second fundamental theorem of asset pricing one can easily solve the pricing of the European options: Let H_T be a financial claim at time-period T. As one will get the value H_T at time T the variable H_T is $\mathcal{F}_T$-measurable. A very natural question is that what is the value of H_T at time $t=0$? Assume that there is no arbitrage in the market and assume that the market is complete. Then by the first theorem there is a martingale measure $\mathbf{Q}$. As the market is complete

$$H_T=\lambda+\sum_{t=1}^{T}\left(S(t)-S(t-1),\theta(t)\right). \tag{14}$$

Observe that as there is no arbitrage the constant λ is unique. If

$$H_T=\lambda_i+\sum_{t=1}^{T}\left(S(t)-S(t-1),\theta_i(t)\right),\quad i=1,2$$

with $\lambda_1 > \lambda_2$ then trivially

$$\sum_{t=1}^{T} \left(S(t) - S(t-1), \theta_2(t) - \theta_1(t)\right) = \lambda_1 - \lambda_2 > 0$$

hence $(\theta_2 - \theta_1)$ is clearly an arbitrage strategy[19]. The only reasonable price of H_T is the amount of money $\pi(H_T)$ which will not introduce an arbitrage to the market. It is clear that we are not introducing an arbitrage if and only if $\pi(H_T) = \lambda$. Of course one can ask that how one can express the value of λ with $\mathbf{Q}$. We can assume[20] that H_T is integrable under the original measure $\mathbf{P}$. As the derivative $d\mathbf{P}/d\mathbf{Q}$ is bounded H_T is integrable under $\mathbf{Q}$. This means that $\sum_{t=1}^{T}(S(t) - S(t-1), \theta(t))$ is a martingale that is

$$\mathbf{E}^{\mathbf{Q}}\left(\sum_{t=1}^{T}(S(t) - S(t-1), \theta(t))\right) = 0.$$

Hence taking an expected value with respect to $\mathbf{Q}$ in line (14)

$$\begin{aligned} \pi(H_T) &= \lambda + 0 = \lambda + \mathbf{E}^{\mathbf{Q}}\left(\sum_{t=1}^{T}(S(t) - S(t-1), \theta(t))\right) = \qquad (15) \\ &= \mathbf{E}^{\mathbf{Q}}(H_T). \end{aligned}$$

One can recall that we were not really using that the market is complete. We only used that for H_T the representation (14) holds. If the market is not complete and (14) holds for some H_T then the pricing formula (15) is valid. Using the no-arbitrage assumption one can easily show that the price $\pi(H_T)$ is independent of $\mathbf{Q}$.

Finally let us make a quite surprising remark:

Proposition 16 (Lost illusions) *If the market is complete and there is no arbitrage then the probability measure* $(\Omega, \mathcal{F}_{\mathbf{T}}, \mathbf{P})$ *is generated by finite number of atoms.*

Proof: Let H_T be an arbitrary $\mathcal{F}_T$-measurable claim. Changing the measure[21] one can assume that H_T is integrable under the original measure $\mathbf{P}$. As the Radon–Nikodym derivative is bounded H_T is integrable under the martingale measure $\mathbf{Q}$. Under the conditions of the proposition every $\mathcal{F}_T$-measurable claim H_T is integrable with respect to the same measure $\mathbf{Q}$. This implies that $(\Omega, \mathcal{F}_T, \mathbf{Q})$ has only finite number of sets which are disjoint and which has positive measure under $\mathbf{Q}$. As $\mathbf{P}$ and $\mathbf{Q}$ are equivalent $(\Omega, \mathcal{F}_{\mathbf{T}}, \mathbf{P})$ has the same property.

. □

[19] Observe that we used the fact that the set of admissible strategies is a linear space. This not true if the time-horizon is infinite as to exclude the doubling strategy one should restrict the set of admissible strategies to a proper cone.

[20] If necessary we can change the measure to make H_T integrable.

[21] The no-arbitrage condition is invariant under equivalent change of measure.

Proposition 17 *If $\mathcal{F}_0$ is the trivial σ-algebra then the number of atoms in $\mathcal{F}_T$ is maximum $(m+1)^T$, where m is the number of stocks and T is the length of the time-period.*

Proof: Let N be the dimension of $L^0(\Omega, \mathcal{F}_\mathbf{T}, \mathbf{P})$. This means that the number of atoms in $\mathcal{F}_T$ is N. If $T = 1$ then the completeness means that every $H_T \in \mathbb{R}^N$ has the representation

$$H_T(\omega) = \lambda + (S_T(\omega), \boldsymbol{\theta}_T(\omega)).$$

As $\mathcal{F}_0$ is just the trivial σ-algebra every random variable with respect to $\mathcal{F}_0$ is deterministic. Hence $\boldsymbol{\theta}_T$ is just an m-dimensional vector. Hence $(S_T, \boldsymbol{\theta}_T)$ is just the set of all possible linear combinations of m vectors, so

$$\{(S_T, \boldsymbol{\theta}_T) : \boldsymbol{\theta}_T \text{ is } \mathcal{F}_{T-1} \text{ measurable}\}$$

has dimension not bigger than m. As we have an extra dimension because of λ if $T = 1$ then the dimension of $L^0(\Omega, \mathcal{F}_\mathbf{T}, \mathbf{P})$ is maximum $m+1$. Assume that the proposition is valid for $T-1$. In this case

$$\dim\left(L^0(\Omega, \mathcal{F}_{T-1}\mathbf{P})\right) \le (m+1)^{T-1},$$

that is the number of atoms in $\mathcal{F}_{T-1}$ is maximum $(m+1)^{T-1}$. Hence the dimension of all possible θ_T is $(m+1)^{T-1}$. Again we can use only $(m+1)$ vectors so in this case $N \le (m+1)^T$.

□

3 Counterexamples

What does happen if we drop the assumptions of the theorem?

Example 18 *One cannot drop the assumption of the finiteness of the time-periods.*

Let the time-horizon $t = 0, 1, 2 \ldots.$ and let Ω be an infinite binary tree, that is let $\Omega \doteq \{-1, 1\}^{\mathbb{N}}$. This means that if $\omega \in \Omega$ then ω is an infinite sequence with values ± 1. Let $S(t, \omega) \doteq \sum_{i=1}^{t} \omega_i$ be the random walk generated by Ω. Assume that at every time-period the probability of $+1$ is p_0 and the probability of -1 is p_1. That is let $\mathbf{P}$ be the infinite product measure generated by the probability measures $\mathbf{P}_i(\{1\}) = p_0$, $\mathbf{P}_i(\{-1\}) = p_1$. It is easy to see that the only martingale measure for S is the product measure generated by $\mathbf{Q}_i(\{1\}) = 1/2$, $\mathbf{Q}_i(\{-1\}) = 1/2$. By Kakutani's theorem[22] for every infinite product measure if $\mathbf{Q}_i \ll \mathbf{P}_i$ then for the two infinite products $\mathbf{P}$ and $\mathbf{Q}$ either $\mathbf{Q} \ll \mathbf{P}$ or $\mathbf{Q} \perp \mathbf{P}$. One can also show[23] that if the components of the products

[22] See: [15] Theorem 3, page 528.
[23] See: [15] Problem 6.7 page 536.

has the same distribution that is if $\mathbf{P}_i = \mathbf{P}_j$ and $\mathbf{Q}_i = \mathbf{Q}_j$ for every i and j then if $\mathbf{P}_i \neq \mathbf{Q}_i$ then $\mathbf{Q} \perp \mathbf{P}$. This means that if $p_0 \neq 1/2$ then the martingale measure $\mathbf{Q}$ is not equivalent with the original measure $\mathbf{P}$.

One can show this without referring to Kakutani's theorem. One can observe that our measure spaces $\mathbf{P}$ and $\mathbf{Q}$ have a simple representation on the interval $[0,1]$. At the first step we divide the interval into two parts. We choose the first part $[0,1/2]$ with probability p_0 and the interval $[1/2,1]$ with probability $p_1 \circeq 1 - p_0$. At the second step we split the chosen interval into two intervals with the same length and again choose the lower part with probability p_0 and the upper part with probability p_1. With this procedure after infinite number of steps finally we choose a $\xi(\omega)$. If $p_0 = p_1 = 1/2$ then the distribution of ξ is Lebesgue's measure. We show that if $p_0 \neq 1/2$ then the distribution of ξ is singular to Lebesgue's measure. Let ξ_k be 0 or 1 depending we have chosen the lower or the upper interval. Let (u_k) be a sequence with values 0 and 1. Then by the independence of variables ξ_k

$$\mathbf{P}(\xi_k = u_k,\ k = 1,2,\ldots) = \prod_{k=1}^{\infty} p_{u_k} \leq \lim_{n\to\infty} (\max(p_0,p_1))^n = 0.$$

As for every $x \in [0,1]$ there is maximum two sequences (u_k) for which $x = \sum_{k=1}^{\infty} u_k 2^{-k}$, therefore $\mathbf{P}(\xi = x) = 0$. Hence the distribution function

$$F(x) = \mathbf{P}(\xi < x)$$

is continuous. F is an increasing function so it is almost surely differentiable. We show that $F'(x) = 0$ almost surely with respect to Lebesgue's measure. If F is differentiable at x then whenever $u, v \to 0$ then

$$\frac{F(x+u) - F(x-v)}{u-v} \to F'(x).$$

Let (k_n) be a sequence for which $x \in I_n \circeq [k_n/2^n, (k_n+1)/2^n]$.

$$\frac{\mathbf{P}(\xi \in I_n)}{2^{-n}} = \frac{F((k_n+1)/2^n) - F(k_n/2^n)}{2^{-n}} \to F'(x).$$

If $F'(x) \neq 0$ then

$$\frac{\mathbf{P}(\xi \in I_{n+1})}{2^{-n-1}} \Big/ \frac{\mathbf{P}(\xi \in I_n)}{2^{-n}} \to 1,$$

that is

$$\frac{\mathbf{P}(\xi \in I_{n+1})}{\mathbf{P}(\xi \in I_n)} \to \frac{1}{2}. \tag{16}$$

One gets the event $\{\xi \in I_{n+1}\}$ in such a way that we split the interval I_n into two parts and choose the lower part with probability p_0. Hence using the independence of the steps the ratio in (16) is either p_0 or p_1. As $p_0 \neq p_1$ (16) is impossible. Hence the $\mathbf{P} \ll \mathbf{Q}$ is impossible, since if $\mathbf{P} \ll \mathbf{Q}$ then

$$F(x) = \int_0^x \frac{d\mathbf{P}}{d\mathbf{Q}} d\mathbf{Q} = \int_0^x \frac{d\mathbf{P}}{d\mathbf{Q}}(t)\, dt$$

and therefore $\frac{d\mathbf{Q}}{d\mathbf{P}}=0$ almost surely with respect to $\mathbf{Q}$ which is impossible.

Now we show that in an infinite binary tree there is no arbitrage. Of course one should clearly define what is the exact meaning of the no-arbitrage condition. That is what type of trading strategies are allowed.

1. Assume first that "every" trading strategy is allowed that is let assume that every series of predictable strategies (θ_n) is a possible trading strategy. That is let assume that every "stochastic integral"

$$V_n \circeq \sum_{k=1}^{n} \theta_k \left(S_k - S_{k-1}\right)$$

is a legitimate result of some trading activity. In this case with the doubling strategy one can obviously make an arbitrage.

2. To exclude the doubling strategy one should perhaps require that the net gain process (V_n) is bounded from below[24]. As we have a martingale measure, the measure for which the probability of moving up or down is $\pm 1/2$, the series (V_n) is a martingale transform and therefore (V_n) is a local martingale[25]. But as it is bounded from below it is a supermartingale so its expected value is decreasing. Hence $\mathbf{E}^{\mathbf{Q}}\left(V_\tau\right) \le \mathbf{E}^{\mathbf{Q}}\left(V_0\right)=0$ for every stopping time τ. Therefore if $H_\tau \ge 0$ then $H_{\tau.} \overset{a.s.}{=} 0$ with respect to $\mathbf{Q}$. Observe that as V is a supermartingale with respect to $\mathbf{Q}$ it is almost surely convergent under $\mathbf{Q}$ so $\mathbf{E}^{\mathbf{Q}}\left(V\left(\infty\right)\right)=0$. But what happens under $\mathbf{P}$? It can happen that V is almost surely not convergent under $\mathbf{P}$. But how then one can "take the money out" under $\mathbf{P}$? As $V\left(\infty\right)$ is not necessarily exists under $\mathbf{P}$ the only reasonable way is to "take the money out" is to take it at some finite stopping time τ. By definition of the stopping time $\{\tau = n\} \in \mathcal{F}_n$ for every n. By the structure of the generated σ-algebras $\mathbf{P}\left(\{\tau=n\}\right)>0$ and $\mathbf{Q}\left(\{\tau=n\}\right)>0$ so as τ is finite

$$\Omega = \{\tau < \infty\} = \cup_n \{\tau = n\}.$$

so $V_\tau = 0$ almost surely under $\mathbf{P}$ and $\mathbf{Q}$.

3. On can say that in the above example the restriction of the trading strategies to the strategies with net payoff bounded from below is unnecessary, or too strong. One can exclude the doubling strategy if one is not allowing to change the position arbitrary many times[26]. A weak restriction of this type is when one allows only the so-called simple trading strategies. By definition a trading strategy θ is a simple trading strategy if

$$\theta = \sum_{i=0}^{n-1} \rho_i \chi\left(\left(\tau_i, \tau_{i+1}\right]\right) \tag{17}$$

[24] This implies that the possible trading strategies are not forming a linear space only a cone.

[25] Of course under the martingale measure $\mathbf{Q}$.

[26] This happens when we stop V at some finite stopping time.

where $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_n < \infty$ are stopping times and ρ_i is $\mathcal{F}_{\tau_i}$-measurable random variables for every i. In this case the net result of the trade is

$$V_n = \sum_{i=0}^{n-1} \rho_i \left(S\left(\tau_{i+1}\right) - S\left(\tau_i\right)\right). \tag{18}$$

First we remark that if there is no arbitrage using simple trading strategies (17) then there is no arbitrage strategy of the form

$$\theta = \rho\chi\left((\sigma, \tau]\right) \tag{19}$$

where of course $\sigma \leq \tau < \infty$ and σ and τ are stopping times and ρ is an $\mathcal{F}_\sigma$-measurable random variable. We shall call the strategies of type (17) supersimple trading strategies. The proof proceeds by induction on the number of strategies in (17). If $n = 1$ then the observation is trivial as in this case (17) is the same as (19). Assume that the observation is valid for some n. Let us take a simple arbitrage strategy with $n+1$ components. If there is an arbitrage already with the first n component then by the induction hypotheses there is a supersimple arbitrage strategy. So one can assume that in (18) either $V_n = 0$ or $V_n < 0$ on a set with positive probability. If $V_n = 0$ then

$$\rho_n \left(S\left(\tau_{n+1}\right) - S\left(\tau_n\right)\right) = \sum_{i=0}^{n} \rho_i \left(S\left(\tau_{i+1}\right) - S\left(\tau_i\right)\right) = V_{n+1}$$

is a supersimple arbitrage strategy. If $A \circeq \{V_n < 0\} \in \mathcal{F}_{\tau_n}$ has positive measure then as $V_{n+1} \geq 0$ the supersimple trading strategy $\rho_n\chi_A\chi\left((\tau_n, \tau_{n+1}]\right)$ is an arbitrage strategy. This also shows that if there is an arbitrage with simple strategies then there is a supersimple arbitrage strategy with $|\rho| = 1$, that is $\rho \circeq \operatorname{sign}\left(S\left(\tau\right) - S\left(\sigma\right)\right)$. So if S is bounded[27] then the martingale transform $\rho\left(S\left(\tau\right) - S\left(\sigma\right)\right)$ is bounded and for the same reason as above there is no arbitrage under $\mathbf{Q}$ and as $t < \infty$ under $\mathbf{P}$.

□

One can ask that why in the Kreps–Yan theorem one should assume that $L_-^1 \subseteq K$. With a counterexample[28] we show that the theorem is not valid if we drop the condition $L_-^1 \subseteq K$.

Example 19 *The Kreps–Yan theorem is not valid if one drops the assumption that* $L_-^1 \subseteq C$.

Let (A_n) be a partition of Ω with $\mathbf{P}\left(A_n\right) = 2^{-n}$. Let us also split A_n into A_n^+ and A_n^- with $\mathbf{P}\left(A_n^{\pm}\right) = 2^{-(n+1)}$. Let $B_n \circeq \cup_{k=n+1}^{\infty} A_k$. Let

$$f_n \circeq 2^{3n}\chi_{A_n^+} + 2^n\chi_{A_n^-} - 2^{-n}\chi_{B_n}.$$

[27] To make S bounded one should define a binary tree with jumps size not ± 1 but $\pm (1/2)^n$.
[28] See: [14].

The "matrix" of the functions f_n is

$$\begin{pmatrix} & f_1 & f_2 & f_3 & \cdots \\ A_1^+ & 2^3 & 0 & 0 & \cdots \\ A_1^- & 2^1 & 0 & 0 & \cdots \\ A_2^+ & -1/2 & 2^6 & 0 & \cdots \\ A_2^- & -1/2 & 2^2 & 0 & \cdots \\ A_3^+ & -1/2 & -1/2^2 & 2^9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let

$$L \circeq \operatorname{lin}(f_n : n \in \mathbb{N}) \quad \text{and} \quad K \circeq \operatorname{cl}(L),$$

where we take the closure of the linear subspace L with respect to the topology of L^1. Obviously K is a closed cone in L^1.

We show that K satisfies the no-arbitrage condition. Let $g_0 \in K \cap L^1_+$. From the construction it is clear that every function in L is constant on every set $A_n^{\pm}$. As convergence in L^1 implies the almost sure convergence for a subsequence the same relation is true for every $g \in K$, that is for g_0. Let n_0 be the first index where g_0 is not almost surely zero on A_{n_0}. If $l_n \overset{a.s.}{\to} g_0$ for some $(l_n) \in L$ then

$$l_n \chi\left(\cup_{k=1}^{n_0-1}\right) \overset{a.s.}{\to} 0.$$

By the construction it is clear that if $n > n_0$ then f_n is zero on A_{n_0}. From the "matrix" it is clear that the coordinates of f_n with $n < n_0$ goes to zero. So there is an $a > 0$ such that g_0 is $a2^{3n_0}$ on $A_{n_0}^+$ and $a2^{n_0}$ on $A_{n_0}^-$. Obviously the coordinates $\left(\lambda_{n_0}^{(n)}\right)$ of f_{n_0} converge to $a > 0$. By definition f_{n_0} is -2^{-n_0} on A_{n_0+1} and if $k \geq n_0 + 2$ then f_k is zero on $A_{n_{0+1}}$, so

$$l_n \chi(A_{n_0+1}) = \left(\lambda_{n_0}^{(n)} f_{n_0} + \lambda_{n_0+1}^{(n)} f_{n_0+1}\right) \chi(A_{n_0+1}).$$

As this limit is non-negative and as $f_{n_0} < 0$ on A_{n_0+1} obviously one can assume that $\lambda_{n_0+1}^{(n)} > 0$ and $\lambda_{n_0+1}^{(n)} \to b$. Using the definition of f_{n_0+1} on $A_{n_0+1}^-$ and the condition $g_0 \geq 0$

$$a2^{-n_0} \leq b2^{n_0+1}.$$

Hence on $A_{n_0+1}^+$

$$\begin{aligned} g_0 &= b2^{3(n_0+1)} - a2^{-n_0} \geq \\ &\geq b2^{n_0+1}2^{2(n_0+1)} - b2^{n_0+1} = \\ &= b2^{n_0+1}\left(2^{2(n_0+1)} - 1\right) \geq \\ &\geq a2^{-n_0}\left(2^{2(n_0+1)} - 1\right) \geq \\ &\geq a\left(2^{n_0+2} - 1\right) \geq a2^{2(n_0+1)}. \end{aligned}$$

Now let us concentrate on the set A_{n_0+2}. Here the "negative" part is

$$a2^{-n_0} + b2^{-(n_0+1)} \leq c2^{n_0+2}.$$

On $A^+_{n_0+2}$

$$\begin{aligned} g_0 &= c2^{3(n_0+2)} - b2^{-(n_0+1)} - a2^{-n_0} \geq \\ &\geq c2^{3(n_0+2)} - c2^{n_0+2} = \\ &= c2^{n_0+2}\left(2^{2(n_0+2)} - 1\right) \geq \\ &\geq a2^{-n_0}\left(2^{2n_0+4} - 1\right) \geq a2^{n_0+2}. \end{aligned}$$

Continuing one can show that on $A^+_{n_0+k}$

$$g_0 \geq a \cdot 2^{n_0+k}.$$

As $\mathbf{P}\left(A^+_k\right) = 2^{-(k+1)}$

$$\mathbf{E}\left(g_0\right) \geq a \sum_{k=n_0}^{\infty} \frac{2^k}{2^{k+1}} = \infty$$

which is impossible as $g_0 \in L^1$. So K satisfies the no-arbitrage assumption.

On the other hand if $g_n = \sum_{k=1}^{n} f_k$ then $g_n \geq -1$ and $g_n \geq 1$ on the set B^c_n. Therefore $h_n \circeq g_n \wedge 1 \in K - L^1_+$ and obviously $|h_n| \leq 1$. As $\mathbf{P}\left(B_n\right) \to 0$ from the Dominated Convergence Theorem

$$\lim_{n\to\infty} h_n = 1$$

in L^1. Now if the Kreps–Yan theorem were true for K then there was a non-negative and not zero function $z \in L^\infty_+$ such that $\langle z, K\rangle \leq 0$. As $z \geq 0$ obviously

$$\langle z, h_n\rangle \leq 0.$$

But $h_n \to 1$ in L^1 so $\langle z, 1\rangle \leq 0$ which is impossible.

□

Example 20 *There is a closed linear space K in the Banach space L^1 such that*

$$K \cap L^1_+ = \{0\}$$

and the convex cone $K - L^1_+$ is not closed.

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