CREDIT SUISSE | FIRST BOSTON

CreditRisk^{+}

A CREDIT RISK MANAGEMENT FRAMEWORK

Regulated by SFA

CREDIT SUISSE | FIRST BOSTON is a leading global investment banking firm, providing comprehensive financial advisory, capital raising, sales and trading, and financial products for users and suppliers of capital around the world. It operates in over 60 offices across more than 30 countries and six continents and has over 15,000 employees.

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Introduction to CreditRisk+

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1.1 Developments in Credit Risk Management

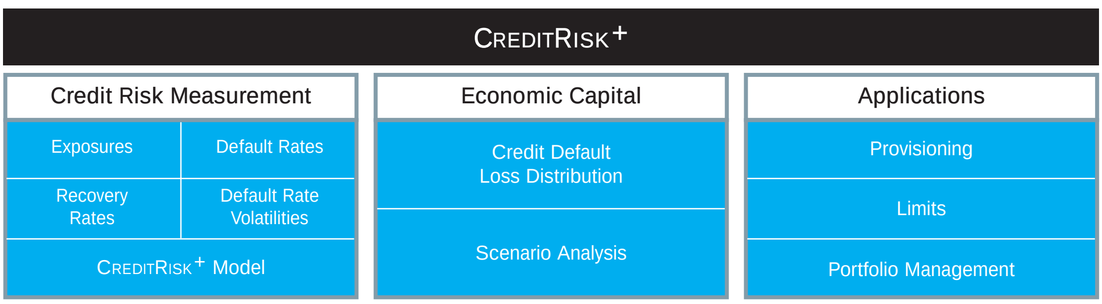
Since the beginning of the 1990s, Credit Suisse First Boston ("CSFB") has been developing and deploying new risk management methods. In 1993, Credit Suisse Group launched, in parallel, a major project aimed at modernising its credit risk management and, using CSFB's expertise, at developing a more forward-looking management tool. In December 1996, Credit Suisse Group introduced CreditRisk+ - a Credit Risk Management Framework.

Current areas of development in credit risk management include: modelling credit risk on a portfolio basis; credit risk provisioning; active portfolio management; credit derivatives; and sophisticated approaches to capital allocation that more closely reflect economic risk than the existing regulatory capital regime. CreditRisk+ addresses all of these areas and the relationships between them.

CreditRisk+ can be applied to credit exposures arising from all types of products including corporate and retail loans, derivatives, and traded bonds.

1.2 Components of CreditRisk+

The components of CreditRisk+ and the interrelationships between them are shown in the following diagram.



◀ Figure 1:
Components of CreditRisk+

CreditRisk+ comprises three main components – a CreditRisk+ Model that uses a portfolio approach, a methodology for calculating economic capital for credit risk, and several applications of the technology.

A modern approach to credit risk management should address all aspects of credit risk, from quantitative modelling to the development of practical techniques for its management. In addition to well-established credit risk management techniques, such as individual obligor (borrower, counterparty or issuer) limits and concentration limits, CreditRisk+ reflects the requirements of a modern approach to managing credit risk and comprises three main components:

- The CreditRisk+ Model that uses a portfolio approach and analytical techniques applied widely in the insurance industry.
- A methodology for calculating economic capital for credit risk.
- Applications of the credit risk modelling methodology including: (i) a methodology for establishing provisions on an anticipatory basis, and (ii) a means of measuring diversification and concentration to assist in portfolio management.

1.3 The CreditRisk+ Model

CreditRisk+ is based on a portfolio approach to modelling credit default risk that takes into account information relating to size and maturity of an exposure and the credit quality and systematic risk of an obligor.

The CreditRisk+ Model is a statistical model of credit default risk that makes no assumptions about the causes of default. This approach is similar to that taken in market risk management, where no attempt is made to model the causes of market price movements. The CreditRisk+ Model considers default rates as continuous random variables and incorporates the volatility of default rates in order to capture the uncertainty in the level of default rates. Often, background factors, such as the state of the economy, may cause the incidence of defaults to be correlated, even though there is no causal link between them. The effects of these background factors are incorporated into the CreditRisk+ Model through the use of default rate volatilities and sector analysis rather than using default correlations as explicit inputs into the model.

Mathematical techniques applied widely in the insurance industry are used to model the sudden event of an obligor default. This approach contrasts with the mathematical techniques typically used in finance. In financial modelling one is usually concerned with modelling continuous price changes rather than sudden events. Applying insurance modelling techniques, the analytic CreditRisk+ Model captures the essential characteristics of credit default events and allows explicit calculation of a full loss distribution for a portfolio of credit exposures.

1.4 Economic Capital

The output of the CreditRisk+ Model can be used to determine the level of economic capital required to cover the risk of unexpected credit default losses.

Measuring the uncertainty or variability of loss and the relative likelihood of the possible levels of unexpected losses in a portfolio of credit exposures is fundamental to the effective management of credit risk. Economic capital provides a measure of the risk being taken by a firm and has several benefits: it is a more appropriate risk measure than that specified under the current regulatory regime; it measures economic risk on a portfolio basis and takes account of diversification and concentration; and, since economic capital reflects the changing risk of a portfolio, it can be used for portfolio management.

The CreditRisk+ Model is supplemented by scenario analysis in order to identify the financial impact of low probability but nevertheless plausible events that may not be captured by a statistically based model.

1.5 Applications of CreditRisk+

CreditRisk+ includes several applications of the credit risk modelling methodology, including a forward-looking provisioning methodology and quantitative portfolio management techniques.

1.6 Example Spreadsheet Implementation

In order to assist the reader of this document, a spreadsheet-based implementation that illustrates the range of possible outputs of the CreditRisk+ Model can be downloaded from the Internet (http://www.csfb.com).

Modelling Credit Risk

2

2.1 Risk Modelling Concepts

2.1.1 Types of Uncertainty Arising in the Modelling Process

A statistically based model can describe many business processes. However, any model is only a representation of the real world. In the modelling process, there are three types of uncertainty that must be assessed: process risk, parameter uncertainty and model error.

Process Risk

Process risk arises because the actual observed results are subject to random fluctuations even where the model describing the loss process and the parameters used by the model are appropriate. Process risk is usually addressed by expressing the model results to an appropriately high level of confidence.

Parameter Uncertainty

Parameter uncertainty arises from the difficulties in obtaining estimates of the parameters used in the model. The only information that can be obtained about the underlying process is obtained by observing the results that it has generated in the past. It is possible to assess the impact of parameter uncertainty by performing sensitivity analysis on the parameter inputs.

Model Error

Model error arises because the proposed model does not correctly reflect the actual process - alternative models could produce different results. Model error is usually the least tractable of the three types of uncertainty.

2.1.2 Addressing Modelling Issues

As all of these types of uncertainty enter into the modelling process, it is important to be aware of them and to consider how they can be addressed when developing a credit risk model. Indeed, a realistic assessment of the potential effects of these errors should be made before any decisions are made based on the outputs of the model.

CreditRisk+ addresses these types of uncertainty in several ways:

- No assumptions are made about the causes of default. This approach is similar to that taken in market risk management, where no assumptions are made about the causes of market price movements. This not only reduces the potential model error but also leads to the development of an analytically tractable model.
- The data requirements for the CreditRisk+ Model have been kept as low as possible, which minimises the error from parameter uncertainty. In the credit environment, empirical data is sparse and difficult to obtain. Even then, the data can be subject to large fluctuations year on year.
- Concerns about parameter uncertainty are addressed using scenario analysis, in which the effects of stress testing each of the input parameters are quantified. For example, increasing default rates or default rate volatilities can be used to simulate downturns in the economy.

■ The CreditRisk+ Model makes no assumptions about the causes of default. This approach is similar to that taken in market risk management, where no assumptions are made about the causes of market price movements.

2.2 Types of Credit Risk

There are two main types of credit risk:

- Credit spread risk: Credit spread risk is exhibited by portfolios for which the credit spread is traded and marked-to-market. Changes in observed credit spreads impact the value of these portfolios.
- Credit default risk: All portfolios of exposures exhibit credit default risk, as the default of an obligor results in a loss.

2.2.1 Credit Spread Risk

Credit spread is the excess return demanded by the market for assuming a certain credit exposure. Credit spread risk is the risk of financial loss owing to changes in the level of credit spreads used in the mark-to-market of a product.

Credit spread risk fits more naturally within a market risk management framework. In order to manage credit spread risk, a firm's value-at-risk model should take account of value changes caused by the volatility of credit spreads. Since the distribution of credit spreads may not be normal, a standard variance-covariance approach to measuring credit spread risk may be inappropriate. However, the historical simulation approach, which does not make any assumptions about the underlying distribution, used in combination with other techniques, provides a suitable alternative.

Credit spread risk is only exhibited when a mark-to-market accounting policy is applied, such as for portfolios of bonds and credit derivatives. In practice, some types of products, such as corporate or retail loans, are typically accounted for on an accruals basis. A mark-to-market accounting policy would have to be applied to these products in order to recognise the credit spread risk.

2.2.2 Credit Default Risk

Credit default risk is the risk that an obligor is unable to meet its financial obligations. In the event of a default of an obligor, a firm generally incurs a loss equal to the amount owed by the obligor less a recovery amount which the firm recovers as a result of foreclosure, liquidation or restructuring of the defaulted obligor. All portfolios of exposures exhibit credit default risk, as the default of an obligor results in a loss.

■ All portfolios of exposures exhibit credit default risk, as the default of an obligor results in a loss.

Credit default risk is typically associated with exposures that are more likely to be held to maturity, such as corporate and retail loans and exposures arising from derivative portfolios. Bond markets are generally more liquid than loan markets and therefore bond positions can be adjusted over a shorter time frame. However, where the intention is to maintain a bond portfolio over a longer time frame, even though the individual constituents of the portfolio may change, it is equally important to measure the default risk that is taken by holding the portfolio.

CreditRisk+ focuses on modelling and managing credit default risk.

2.3 Default Rate Behaviour

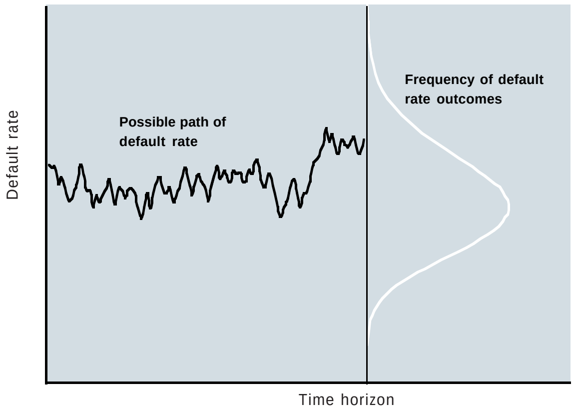
Equity and bond prices are forward-looking in nature and are formed by investors' views of the financial prospects of a particular obligor. Hence, they incorporate both the credit quality and the potential credit quality changes of that obligor.

Therefore, the default rate of a particular obligor, inferred from market prices, will vary on a continuous scale and hence can be viewed as a continuous random variable. In modelling credit risk, one is concerned with determining the possible future outcomes over the chosen time horizon.

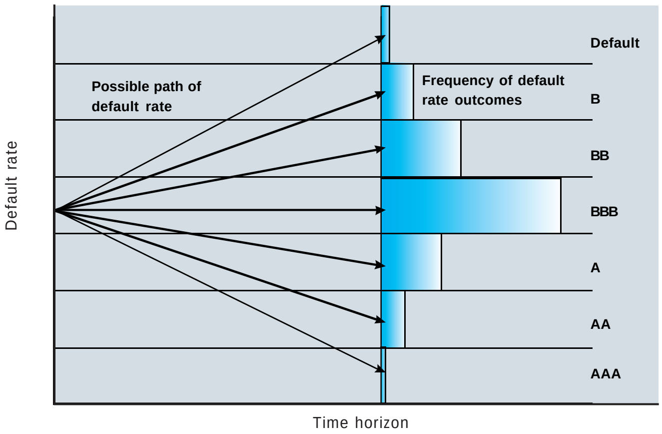
The process for the default rate can be represented in two different ways:

- Continuous variable: When treated as a continuous variable, the possible default rate over a given time horizon is described by a distribution, which can be specified by a default rate and a volatility of the default rate. The data requirements for modelling credit default risk are analogous to the data requirements for pricing stock options - a forward stock price and the stock price volatility are used to define the forward stock price distribution. The following figure illustrates the path that a default rate may take over time and the distribution that it could have over that time.

Figure 2:
Default rate as a continuous random variable



- Discrete variable: By treating the default rate as a discrete variable, a simplification of the continuous process described above is made. A convenient way of making default rates discrete is by assigning credit ratings to obligors and mapping default rates to credit ratings. Using this approach, additional information is required in order to model the possible future outcomes of the default rate. This can be achieved via a rating transition matrix that specifies the probability of keeping the same credit rating, and hence the same value for the default rate, and the probabilities of moving to different credit ratings and hence to different values for the default rate. This is illustrated in the following figure.



◀ Figure 3:
Default rate as a discrete random variable

The discrete approach with rating migrations and the continuous approach with a default rate volatility are different representations of the behaviour of default rates. Both approaches achieve the desired end result of producing a distribution for the default rate.

The above two representations of default rate behaviour are summarised in the following table:

Treatment of default rate	Data requirements
Continuous variable	• Default rates • Volatility of default rates
Discrete variable	• Credit ratings • Rating transition matrix

◀ Table 1:
Representations of the default rate process

■ The CreditRisk+ Model treats default rates as continuous random variables and incorporates default rate volatility to capture the uncertainty in the level of the default rate.

The CreditRisk+ Model is a statistical model of credit default risk that models default rates as continuous random variables and incorporates the volatility of the default rate in order to capture the uncertainty in the level of the default rate. A mapping from credit ratings to a set of default rates provides a convenient approach to setting the level of the default rate.

2.4 Modelling Approach

2.4.1 Risk Measures

When managing credit risk, there are several measures of risk that are of interest, including the following:

- Distribution of loss: The risk manager is interested in obtaining distributions of loss that may arise from the current portfolio. The risk manager needs to answer questions such as “What is the size of loss for a given confidence level?”.
- Identifying extreme outcomes: The risk manager is also concerned with identifying extreme or catastrophic outcomes. These outcomes are usually difficult to model statistically but can be addressed through the use of scenario analysis and concentration limits.

2.4.2 A Portfolio Approach to Managing Credit Risk

Credit risk can be managed through diversification because the number of individual risks in a portfolio of exposures is usually large. Currently, the primary technique for controlling credit risk is the use of limit systems, including individual obligor limits to control the size of exposure, tenor limits to control the maximum maturity of exposures to obligors, rating exposure limits to control the amount of exposure to obligors of certain credit ratings, and concentration limits to control concentrations within countries and industry sectors.

■ CreditRisk+ is based on a portfolio approach - summarising information about size, maturity, credit quality and systematic risk into a single measure.

The portfolio risk of a particular exposure is determined by four factors: (i) the size of the exposure, (ii) the maturity of the exposure, (iii) the probability of default of the obligor, and (iv) the systematic or concentration risk of the obligor. Credit limits aim to control risk arising from each of these factors individually. The general effect of this approach, when applied in a well-structured and consistent manner, is to create reasonably well-diversified portfolios. However, these limits do not provide a measure of the diversification and concentration of a portfolio.

In order to manage effectively a portfolio of exposures, a means of measuring diversification and concentration has to be developed. An approach that incorporates size, maturity, credit quality and systematic risk into a single portfolio measure is required. CreditRisk+ takes such an approach.

2.4.3 Modelling Techniques Used in the CreditRisk+ Model

The economic risk of a portfolio of credit exposures is analogous to the economic risk of a portfolio of insurance exposures. In both cases, losses can be suffered from a portfolio containing a large number of individual risks, each with a low probability of occurring. The risk manager is concerned with assessing the frequency of the unexpected events as well as the severity of the losses.

In order to keep model error to a minimum, no assumptions are made about the causes of default. Mathematical techniques applied widely in the insurance industry are used to model the sudden event of an obligor default. In modelling credit default losses one is concerned with sudden events rather than continuous changes. The essential characteristics of credit default events are captured by applying these insurance modelling techniques. This has the additional benefit that it leads to a credit risk model that is analytically tractable and hence not subject to the problems of precision that can arise when using a simulation-based approach. The analytic CreditRisk+ Model allows rapid and explicit calculation of a full loss distribution for a portfolio of credit exposures.

2.5 Time Horizon for Credit Risk Modelling

A key decision that has to be made when modelling credit risk is the choice of time horizon. Generally, the time horizon chosen should not be shorter than the time frame over which risk-mitigating actions can be taken. CreditRisk+ does not prescribe any one particular time horizon but suggests two possible time horizons that can provide management information relevant for credit risk management:

- A constant time horizon, such as one year.
- A hold-to-maturity or run-off time horizon.

2.5.1 Constant Time Horizon

A constant time horizon is relevant, as it allows all exposures to be considered at the same future date. For various reasons, one year is often taken as a suitable time horizon: credit risk mitigating actions can normally be executed within one year, new capital can be raised to replenish capital eroded by actual credit losses during the period, and, furthermore, one year matches the normal accounting period. Given these factors, CreditRisk+ suggests a time horizon of one year for credit risk economic capital.

2.5.2 Hold-to-Maturity Time Horizon

Alternatively, a hold-to-maturity time horizon allows the full term structure of default rates over the lifetime of the exposures to be recognised. This view of the portfolio enables the risk manager to compare exposures of different maturity and credit quality and is an appropriate tool, in addition to the constant time horizon, for portfolio management. The role that the CreditRisk+ Model plays in active portfolio management is discussed later in this document.

A benchmark time horizon of one year can be used for portfolios where there is an intention to maintain exposures for longer than the term of the booked transactions (e.g. traded bond portfolios).

2.6 Data Inputs to Credit Risk Modelling

2.6.1 Data Inputs

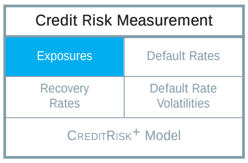
Any modelling of credit risk is dependent on certain data requirements being met. The quality of this data will directly affect the accuracy of the measurement of credit risk and therefore any decision to be made using the results should be made only having fully assessed the potential error from uncertainties in the data used.

The inputs used by the CreditRisk+ Model are:

- Credit exposures
- Obligor default rates
- Obligor default rate volatilities and
- Recovery rates.

The CreditRisk+ Model presented in this document does not prescribe the use of any one particular data set over another. One of the key limitations in modelling credit risk is the lack of comprehensive default data. Where a firm has its own information that is judged to be relevant to its portfolio, this can be used as the input into the model. Alternatively, conservative assumptions can be used while default data quality is being improved.

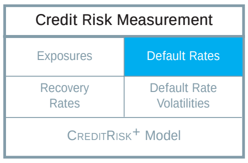
2.6.2 Credit Exposures



The exposures arising from separate transactions with an obligor should be aggregated according to the legal corporate structure and taking into account any rights of set-off.

The CreditRisk+ Model is capable of handling all types of instruments that give rise to credit exposure, including bonds, loans, commitments, financial letters of credit and derivative exposures. For some of these transaction types, it is necessary to make an assumption about the level of exposure in the event of a default: for example, a financial letter of credit will usually be drawn down prior to default and therefore the exposure at risk should be assumed to be the full nominal amount.

In addition, if a multi-year time horizon is being used, it is important that the changing exposures over time are accurately captured.



2.6.3 Default Rates

A default rate, which represents the likelihood of a default event occurring, should be assigned to each obligor. This can be achieved in a number of ways, including:

- Observed credit spreads from traded instruments can be used to provide market-assessed probabilities of default.
- Alternatively, obligor credit ratings, together with a mapping of default rates to credit ratings, provide a convenient way of assigning probabilities of default to obligors. The rating agencies publish historic default statistics by rating category for the population of obligors that they have rated.

▶ Table 2:
One-year default rates (%)

Credit rating	One-year default rate
Aaa	0.00
Aa	0.03
A	0.01
Baa	0.12
Ba	1.36
B	7.27

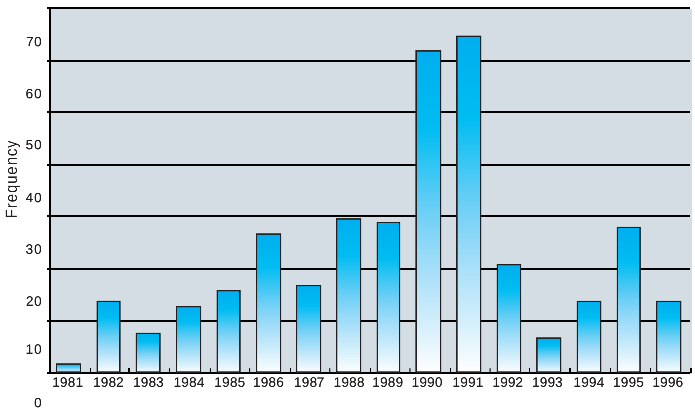
Source: Carty & Lieberman, 1997, Moody's Investors Service Global Credit Research

A credit rating is an opinion of an obligor's overall financial capacity to meet its financial obligations (i.e. its creditworthiness). This opinion focuses on the obligor's capacity and willingness to meet its financial commitments as they fall due. An assessment of the nature of a particular obligation, including its seniority in bankruptcy or liquidation, should be performed when considering the recovery rate for an obligor.

It should be noted that one-year default rates show significant variation year on year, as can be seen in the following figure. During periods of economic recession, the number of defaults can be many times the level observed at other times.

▶ Figure 4:
Rated corporate defaults by number of issuers

■ One-year default rates show significant fluctuations from year to year.



Source: Standard & Poor's Ratings Performance 1996 (February 1997)

- Another approach is to calculate default probabilities on a continuous scale, which can be used as a substitute for the combination of credit ratings and assigned default rates.

2.6.4 Default Rate Volatilities

Credit Risk Measurement
Exposures | Default Rates
Recovery Rates | Default Rate Volatilities
CreditRisk+ Model

Published default statistics include average default rates over many years. As shown previously, actual observed default rates vary from these averages. The amount of variation in default rates about these averages can be described by the volatility (standard deviation) of default rates. As can be seen in the following table, the standard deviation of default rates can be significant compared to actual default rates, reflecting the high fluctuations observed during economic cycles.

Credit rating	One-year default rate (%)	
	Average	Standard deviation
Aaa	0.00	0.0
Aa	0.03	0.1
A	0.01	0.0
Baa	0.12	0.3
Ba	1.36	1.3
B	7.27	5.1

◀ Table 3:
Default rate standard deviations (%)

Source: Carty & Lieberman, 1996, Moody's Investors Service Global Credit Research

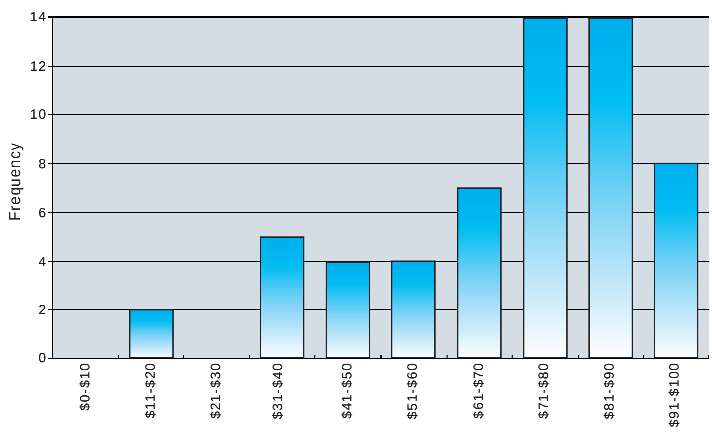
The default rate standard deviations in the above table were calculated over the period from 1970 to 1996 and therefore include the effect of economic cycles.

2.6.5 Recovery Rates

Credit Risk Measurement
Exposures | Default Rates
Recovery Rates | Default Rate Volatilities
CreditRisk+ Model

In the event of a default of an obligor, a firm generally incurs a loss equal to the amount owed by the obligor less a recovery amount, which the firm recovers as a result of foreclosure, liquidation or restructuring of the defaulted obligor or the sale of the claim. Recovery rates should take account of the seniority of the obligation and any collateral or security held.

Recovery rates are subject to significant variation. For example, the figure below shows the price distribution of defaulted bank loans and illustrates that there is a large degree of dispersion.



◀ Figure 5:
Defaulted bank loan price distribution

Source: Defaulted Bank Loan Recoveries (November 1996) , Moody's Investors Service Global Credit Research

There is also considerable variation for obligations of differing seniority, as can be seen from the standard deviation of the corporate bond and bank loan recovery rates in the table below.

Table 4:
Recovery rates by seniority and security (%)

Seniority and security	Average	Standard deviation
Senior secured bank loans	71.18	21.09
Senior secured public debt	63.45	26.21
Senior unsecured public debt	47.54	26.29
Senior subordinated public debt	38.28	24.74
Subordinated public debt	28.29	20.09
Junior subordinated public debt	14.66	8.67

Source: Historical Default Rates of Corporate Bond Issuers, 1920-1996 (January 1997) Moody's Investors Service Global Credit Research

Publicly available recovery rate data indicates that there can be significant variation in the level of loss, given the default of an obligor. Therefore, a careful assessment of recovery rate assumptions is required. Given this uncertainty, stress testing should be performed on the recovery rates in order to calculate the potential loss distributions under different scenarios.

2.7 Correlation and Incorporating the Effects of Background Factors

Default correlation impacts the variability of default losses from a portfolio of credit exposures. The CreditRisk+ Model incorporates the effects of default correlations by using default rate volatilities and sector analysis.

2.7.1 The Random Nature of Defaults and the Appearance of Correlation

■ Often, there are background factors that may cause the incidence of defaults to be correlated, even though there is no causal link between them.

Credit defaults occur as a sequence of events in such a way that it is not possible to forecast the exact time of occurrence of any one default or the exact total number of defaults. Often, there are background factors that may cause the incidence of default events to be correlated, even though there is no causal link between them. For example, if there is an unusually large number of defaults in one particular month, this might be due to the economy being in recession, which has increased the rates of default above their average level. In this economic situation, it is quite likely that the number of defaults in the following month will also be high. Conversely, if there are fewer defaults than on average in one month, because the economy is growing, it is also likely that there will be fewer defaults than on average in the following month. The defaults are correlated but there is no causal link between them - the correlation effect observed is due to a background factor, the state of the economy, which changes the rates of default.

2.7.2 Impact of the Economy on Default Rates

There is general agreement that the state of the economy in a country has a direct impact on observed default rates. A recent report by Standard and Poor's stated that "*A healthy economy in 1996 contributed to a significant decline in the total number of corporate defaults. Compared to 1995, defaults were reduced by one-half…*"[1] Another report by Moody's Investors Service stated that "*The sources of [default rate volatility] are many, but macroeconomic trends are certainly the most influential factors*".[2]

As the above quotations indicate and as can be seen in Figure 4 above, there is significant variation in the number of defaults from year to year. Furthermore, for each year, different industry sectors will be affected to different degrees by the state of the economy. The magnitude of the impact will be dependent on how sensitive an obligor's earnings are to various economic factors, such as the growth rate of the economy and the level of interest rates.

1 Standard and Poor's Ratings Performance 1996, February 1997

2 Moody's Investors Service, Corporate Bond Defaults and Default Rates, January 1996

Economic models that attempt to capture the effect of changes in the economy on default rates can be developed in order to specify the default rates for subsequent use in a credit risk model. However, this approach can have several weaknesses, including the following:

- Since there are limited publicly available default rate statistics by country or by industry sector, it is difficult to verify the accuracy of an economic model used to derive default rates.
- Even if a causal relationship could be established relating default rates to certain economic variables, it is questionable whether such relationships would be stable over several years.

Therefore, alternative approaches that attempt to capture the observed variability of default rates have to be sought.

2.7.3 Incorporating the Effects of Background Factors

It is possible to incorporate the effects of background factors into the specification of default rates by allowing the default rate itself to have a probability distribution. This is accomplished by incorporating default rate volatilities into the model.

The CreditRisk+ Model models the effects of background factors by using default rate volatilities that result in increased defaults rather than by using default correlations as a direct input. Both approaches, the use of default rate volatilities and default correlations, give rise to loss distributions with fat tails.

■ The CreditRisk+ Model captures concentration risk through the use of default rate volatilities and sector analysis.

Section 3 of this document describes in detail how the CreditRisk+ Model uses default rate volatilities in the modelling of credit default risk.

The CreditRisk+ Model does not attempt to model correlations explicitly but captures the same concentration effects through the use of default rate volatilities and sector analysis[3]. There are various reasons why this approach has been taken, including the following:

- Instability of default correlations: Generally, correlations calculated from financial data show a high degree of instability. In addition, a calculated correlation can be very dependent on the underlying time period of the data. A similar instability problem may arise with default rate volatilities: however, it is much easier to perform scenario analysis on default rate volatilities, owing to the analytically tractable nature of a model that uses volatilities rather than correlations.
- Lack of empirical data: There is little empirical data on default correlations. Defaults themselves are infrequent events and so there is insufficient data on multiple defaults with which to calculate explicit default correlations. Since default correlations are difficult to calculate directly, some approaches use asset price correlations to derive default correlations, but this can only be considered a proxy. This technique relies upon additional assumptions about the relationship between asset prices and probabilities of default. Furthermore, it is questionable how stable any relationship, that may be inferred or observed during a period of normal trading, would be in the event of default of a particular obligor. In addition, where there is no asset price for the obligor, for example in a retail portfolio, there is no obvious way of deriving default correlations.

[3] Sector analysis is discussed in Sections 2.8 and 3.4

2.8 Measuring Concentration

The above discussion has highlighted the fact that there are background factors that affect the level of default rates. The state of the economy of each different country will vary over time and, within each country, different industry sectors will be affected to differing degrees. A portfolio of exposures can have concentrations in particular countries or industry sectors. Therefore, it is important to be able to capture the effect of concentration risk in a credit risk model.

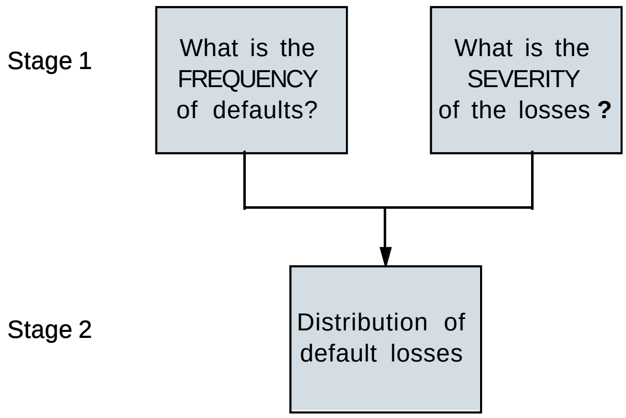
The CreditRisk+ Model described in this document allows concentration risk to be captured using sector analysis. An exposure can be broken down into an obligor-specific element, which is independent of all other exposures, and non-specific or systematic elements that are sensitive to particular driving factors, such as countries or industry sectors.

The CreditRisk+ Model

3

3.1 Stages in the Modelling Process

The modelling of credit risk is a two stage process, as shown in the following diagram:



Credit Risk Measurement	
Exposures	Default Rates
Recovery Rates	Default Rate Volatilities
CreditRisk+ Model	

By calculating the distribution of default events, the risk manager is able to assess whether the overall credit quality of the portfolio is either improving or deteriorating. The distribution of losses allows the risk manager to assess the financial impact of the potential losses as well as measuring the amount of diversification and concentration within the portfolio.

3.2 Frequency of Default Events

3.2.1 The Default Process

The CreditRisk+ Model makes no assumption about the causes of default - credit defaults occur as a sequence of events in such a way that it is neither possible to forecast the exact time of occurrence of any one default nor the exact total number of defaults. There is exposure to default losses from a large number of obligors and the probability of default by any particular obligor is small. This situation is well represented by the Poisson distribution.

We consider first the distribution of the number of default events in a time period, such as one year, within a portfolio of obligors having a range of different annual probabilities of default. The annual probability of default of each obligor can be conveniently determined by its credit rating and a mapping between default rates and credit ratings. If we do not incorporate the volatility of the default rate, the distribution of the number of default events will be closely approximated by the Poisson distribution. This is regardless of the individual default rate for a particular obligor.

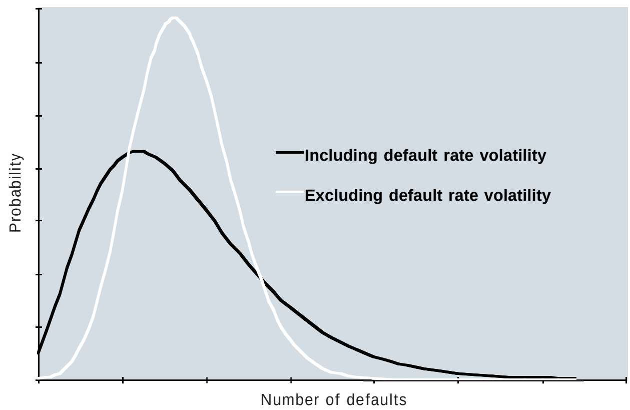
However, default rates are not constant over time and, as we have seen in the previous section, exhibit a high degree of variation. Hence, default rate variability needs to be incorporated into the model.

3.2.2 Distribution of the Number of Default Events

The CreditRisk+ Model models the underlying default rates by specifying a default rate and a default rate volatility. This aims to take account of the variation in default rates in a pragmatic manner, without introducing significant model error.

The effect of using default rate volatilities can be clearly seen in the following figure, which shows the distribution of the number of default events generated by the CreditRisk+ Model when default rate volatility is varied. Although the expected number of default events is the same, the distribution becomes significantly skewed to the right when default rate volatility is increased. This represents a significantly increased risk of an extreme number of default events.

Figure 6:
CreditRisk+ Model - Distribution of default events



3.3 Moving from Default Events to Default Losses

3.3.1 Distribution of Default Losses

Given the number of default events, we now consider the distribution of losses in the portfolio. The distribution of losses differs from the distribution of default events because the amount lost in a given default depends on the exposure to the individual obligors. Unlike the variation of default probability between obligors, which does not influence the distribution of the total number of defaults, the variation in exposure magnitude results in a loss distribution that is not Poisson in general. Moreover, information about the distribution of different exposures is essential to the overall distribution. However, it is possible to describe the overall distribution of losses because its probability generating function has a simple closed form amenable to computation.

In the event of a default of an obligor, a firm generally incurs a loss equal to the amount owed by the obligor less a recovery amount, which the firm obtains as a result of foreclosure, liquidation or restructuring of the defaulted obligor. A recovery rate is used to quantify the amount received from this process. Recovery rates should take account of the seniority of the obligation and any collateral or security held.

In order to reduce the amount of data to be processed, two steps are first followed:

- The exposures are adjusted by anticipated recovery rates in order to calculate the loss in a given default.
- The exposures, net of the above recovery adjustment, are divided into bands of exposure with the level of exposure in each band being approximated by a common average.

The CREDITRISK+ Model calculates the probability that a loss of a certain multiple of the chosen unit of exposure will occur. This allows a full loss distribution to be generated, as shown in the figure below.

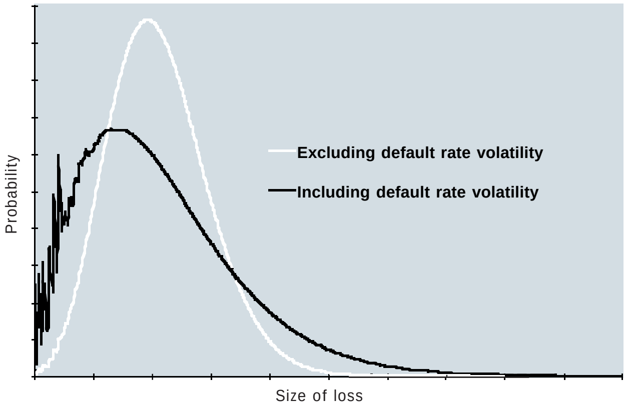


Figure 7: CREDITRISK+ Model - Distribution of default losses

The CREDITRISK+ Model allows explicit calculation of the loss distribution of a portfolio of credit exposures.

3.3.2 Impact of Incorporating Default Rate Volatilities

Figure 7 compares the default loss distributions calculated without default rate volatility and with default rate volatility. The key features and differences are:

- Same expected loss: Both default loss distributions have the same level of expected losses.
- Fatter tail: The key change is the level of losses at the higher percentiles; for example, the 99th percentile is significantly higher when the impact of the variability of default rates is modelled. There is now considerably more chance of experiencing extreme losses.

Since the tail of the distribution has become fatter, while the expected loss has remained unchanged, it can be concluded that the variance of the default loss distribution has increased. This increase in the variance is due to the pairwise default correlations between obligors. These pairwise default correlations are incorporated into the CREDITRISK+ Model through the default rate volatilities and sector analysis. It should be noted that when the default rate volatilities are set to zero, the default events are independent and hence the pairwise default correlations are also zero.

In Appendix A, we give an explicit formula for the pairwise default correlations implied by the CREDITRISK+ Model when default rate volatilities are incorporated into the model.

3.4 Concentration Risk and Sector Analysis

The CreditRisk+ Model measures the benefit of portfolio diversification and the impact of concentrations through the use of sector analysis.

3.4.1 Concentration Risk

■ Concentration risk is dependent on the systematic factors affecting the portfolio. The technique for measuring concentration risk is sector analysis.

Diversification arises naturally because the number of individual risks in a portfolio of exposures is usually large. However, even in a portfolio containing a large number of exposures, there may be an opposing effect owing to concentration risk. Concentration risk results from having in the portfolio a number of obligors whose fortunes are affected by a common factor. In order to quantify concentration risk, the concepts of systematic factors and specific factors are necessary.

Systematic factors

Systematic factors are background factors that affect the fortunes of a proportion of the obligors in the portfolio, for example all those obligors having their domicile in a particular country. The fortunes of any one obligor can be affected by a number of systematic factors.

Specific factors

In general, the fortunes of an obligor are affected to some extent by specific factors unique to the obligor. Systematic factors impact the risk of extreme losses from a portfolio of credit exposures, while diversification largely eliminates the impact of the specific factors.

Concentration risk is dependent on the systematic factors affecting the portfolio. The technique for measuring concentration risk is sector analysis.

3.4.2 Sector Analysis - Allocating all Obligors to a Single Sector

The most straightforward application of the CreditRisk+ Model is to allocate all obligors to a single sector. This approach assumes that a single systematic factor affects the individual default rate volatility of each obligor. Furthermore, this use of the model captures all of the concentration risk within the portfolio and excludes the diversification benefit of the fortunes of individual obligors being subject to a number of independent systematic factors.

Therefore, the most straightforward application of the CreditRisk+ Model, in which all obligors are allocated to a single sector, generates a prudent estimate of extreme losses.

3.4.3 Sector Analysis - Allocating Obligors to one of Several Sectors

In order to recognise some of the diversification benefit of obligors whose fortunes are affected by a number of independent systematic factors, it can be assumed that each obligor is subject to only one systematic factor, which is responsible for all of the uncertainty of the obligor's default rate. For example, obligors could be allocated to sectors according to their country of domicile. Once allocated to a sector, the obligor's default rate and default rate volatility are set individually. In this case, a sector can be thought of as a collection of obligors having the common property that they are influenced by the same single systematic factor.

3.4.4 Sector Analysis - Apportioning Obligors across Several Sectors

A more generalised approach is to assume that the fortunes of an obligor are affected by a number of systematic factors. The CreditRisk+ Model handles this situation by apportioning an obligor across several sectors rather than allocating an obligor to a single sector.

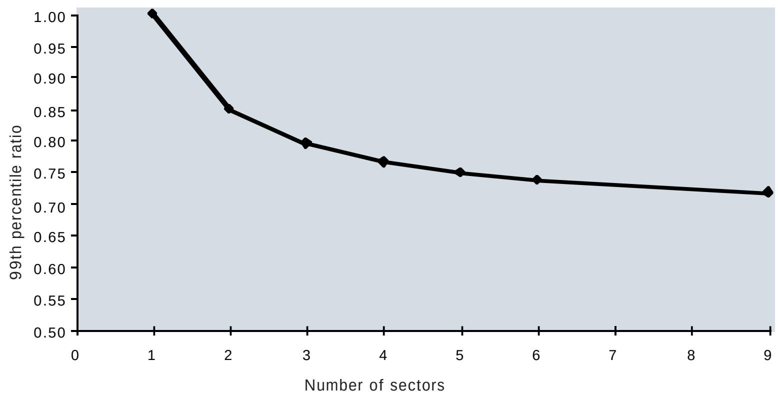
So far it has been assumed that all risk in the portfolio is systematic and allocable to one of the systematic factors. In addition to the effects of systematic factors, it is likely that the fortunes of an obligor are affected by factors specific to the obligor. Potentially specific risk requires an additional sector to model each obligor, since the factor driving specific risk for a given obligor affects that obligor only. However, the CreditRisk+ Model handles specific risk without recourse to a large number of sectors by apportioning all obligors' specific risk to a single "Specific Risk Sector".

3.4.5 The Impact of Sectors on the Loss Distribution

As stated above, the CreditRisk+ Model allows the portfolio of exposures to be allocated to sectors to reflect the degree of diversification and concentration present. The most diversified portfolio is obtained when each exposure is in its own sector and the most concentrated is obtained when the portfolio consists of a single sector.

■ The CreditRisk+ Model allows the portfolio of exposures to be decomposed into sectors to reflect the degree of diversification and concentration present.

The figure below shows the impact of sectors on the loss distribution. As the number of sectors is increased, the impact of concentration risk is reduced. The graph illustrates this by plotting the ratio of the 99th percentile of the credit default loss distribution for a given number of sectors to the 99th percentile of the credit default loss distribution when the portfolio is considered to be a single sector.



◀ Figure 8: Impact of sectors on the loss distribution

3.5 Multi-Year Losses for a Hold-to-Maturity Time Horizon

As discussed in Section 2.5, the CreditRisk+ Model allows risk of the portfolio to be viewed on a hold-to-maturity time horizon in order to capture any default losses that could occur until maturity of the credit exposure.

■ The CreditRisk+ Model allows the risk of the portfolio to be viewed on a hold-to-maturity time horizon in order to capture any default losses that could occur until maturity of the credit exposure.

Analysing credit exposures on a multi-year basis enables the risk manager to compare exposures of different size, credit quality, and maturity. The loss distribution produced provides, for any chosen level of confidence, an indication of the possible cumulative losses that could be suffered until all the exposures have matured. The benefits of looking at portfolio credit risk from this viewpoint include the following:

- The full term structure of default probabilities is taken into account.
- The full uncertainty of default losses over the life of the portfolio is also captured.

For example, because the one-year average default rates for investment grade obligors are relatively small but the corresponding exposures may be large, a one-year time horizon may not be the best measure for active portfolio management. However, a multi-year view will reflect the fact that defaults follow a decline in credit quality over many years.

3.5.1 Using the CreditRisk+ Model to Calculate Multi-Year Loss Distributions

The CreditRisk+ Model can be used to calculate multi-year loss distributions by decomposing the exposure profile over time into separate elements of discrete time, with the present value of the remaining exposure in each time period being assigned a marginal default probability relevant to the maturity and credit quality. These decomposed exposure elements can then be used by the CreditRisk+ Model to generate a loss distribution on a hold-to-maturity basis.

3.6 Summary of the CreditRisk+ Model

The key features of the CreditRisk+ Model are:

- The Credit Risk+ Model captures the essential characteristics of credit default events. Credit default events are rare and occur in a random manner with observed default rates varying significantly from year to year. The approach adopted reflects these characteristics by making no assumptions about the timing or causes of default events and by incorporating the default rate volatility. By taking a portfolio approach, the benefits of diversification that arise from a large number of individual risks are fully captured. Concentration risk, resulting from groups of obligors that are affected by common factors, is measured using sector analysis.
- The Credit Risk+ Model is scaleable and computationally efficient. The CreditRisk+ Model is highly scaleable and hence is capable of handling portfolios containing large numbers of exposures. The low data requirements and minimum of assumptions make the CreditRisk+ Model easy to implement for a wide range of credit risk portfolios, regardless of the specific nature of the obligors. Furthermore, the efficiency of the model allows comprehensive sensitivity analysis to be performed on a continuing basis, which is a key requirement for the ability to quantify the effects of parameter uncertainty.

4 Economic Capital for Credit Risk

4.1 Introduction to Economic Capital

4.1.1 The Role of Economic Capital

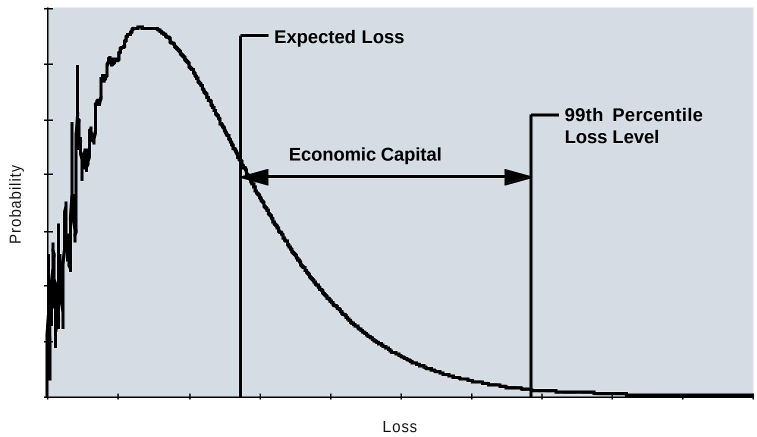
The analysis of uncertainty is the essence of risk management. Therefore, measuring the uncertainty or variability of loss and the related likelihood of the possible levels of unexpected losses in a portfolio of exposures is fundamental to the effective management of credit risk. Sufficient earnings should be generated through adequate pricing and provisioning to absorb any expected loss. The expected loss is one of the costs of transacting business which gives rise to credit risk. However, economic capital is required as a cushion for a firm’s risk of unexpected credit default losses, because the actual level of credit losses suffered in any one period could be significantly higher than the expected level.

4.2 Economic Capital for Credit Risk

4.2.1 Credit Default Loss Distribution

Economic Capital
Credit Default Loss Distribution
Scenario Analysis

Knowledge of the credit default loss distribution arising from a portfolio of exposures provides a firm with management information on the amount of capital that the firm is putting at risk by holding the credit portfolio. Given that economic capital is necessary as a cushion for a firm’s risk of unexpected credit default losses, a percentile level provides a means of determining the level of economic capital for a required level of confidence. In order to capture a significant proportion of the tail of the credit default loss distribution, the 99th percentile unexpected loss level over a one-year time horizon is a suitable definition for credit risk economic capital. This can be seen in the following figure.



▶ Figure 9:
Economic capital for credit risk

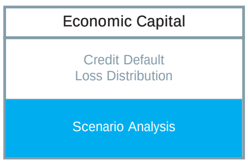
4.2.2 Benefits and Features of Economic Capital

Economic capital as a measure of risk being taken by a firm has several features and benefits including the following:

- It is a more appropriate measure of the economic risk than that specified under the current regulatory regime.
- It measures economic risk on a portfolio basis and hence takes account of the benefits of diversification.
- It is a measure that objectively differentiates between portfolios by taking account of credit quality and size of exposure.
- It is a dynamic measure, which reflects the changing risk of a portfolio and hence can be used as a tool for portfolio optimisation.

4.3 Scenario Analysis

4.3.1 The Role of Scenario Analysis



The purpose of scenario analysis is to identify the financial impact of low probability but nevertheless plausible events that may not be captured by a statistically based model. Therefore, the use of a credit risk model should be supplemented by a programme of stress testing of the assumptions used.

There are two types of stress tests that should be performed: (i) scenario analysis within the CreditRisk+ Model, and (ii) scenario analysis outside the CreditRisk+ Model.

4.3.2 Scenario Analysis within the CreditRisk+ Model

The inputs into the CreditRisk+ Model can be stressed individually or in combination. For example, it is possible to simulate downturns in the economy by increasing default rates and default rate volatilities - sectors of the portfolio can be stressed to varying degrees reflecting the fact that each sector could be affected to a different extent. Similarly, the financial impact of rating downgrades can be assessed by increasing the default rate assigned to an obligor. For a derivatives portfolio, this can be extended to include the effects of movements in market rates on credit exposures.

Given the efficient manner in which the default loss distribution can be calculated, it is possible to calculate the impact of changing parameter inputs used by the model across a wide range of values.

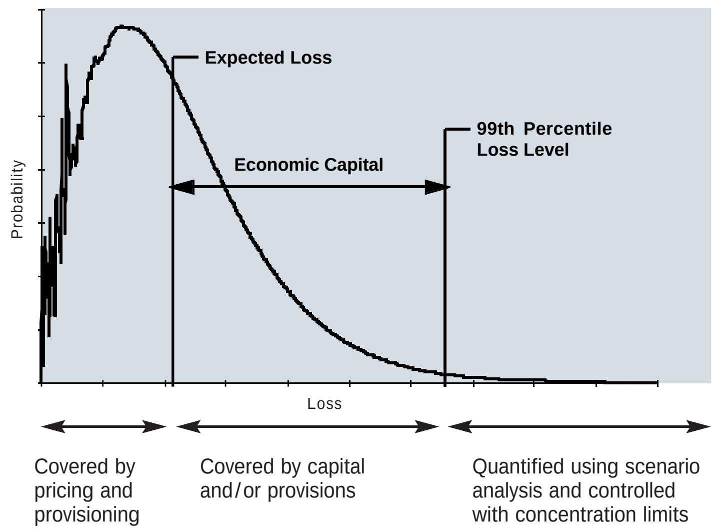
4.3.3 Scenario Analysis outside the CreditRisk+ Model

Certain stress tests can be difficult to perform within the CreditRisk+ Model: for example, the impact of political or financial uncertainty within a country. For these types of scenarios, analysis that is conducted without reference to the outputs of the CreditRisk+ Model, such as looking at the exposure at risk for a given scenario, provides a realistic means of quantifying the financial impact.

■ Scenario analysis provides a means of quantifying catastrophic losses - potential losses can be controlled through concentration limits.

A firm should control the risk of catastrophic losses through the use of obligor and concentration limits, keeping any one of these limits within the loss for the percentile level used to determine the economic capital given by the CreditRisk+ Model.

The figure below illustrates the way in which the distribution of losses can be considered to be divided into three parts.



◀ Figure 10: Parts of the credit default loss distribution

It is possible to control the risk of losses that fall within each of the three parts of the loss distribution in the following ways:

Part of loss distribution	Control mechanism
Up to Expected Loss	Adequate pricing and provisioning
Expected Loss - 99th Percentile Loss	Economic capital and/or provisioning
Greater than 99th Percentile Loss	Quantified using scenario analysis and controlled with concentration limits

◀ Table 5: Mechanisms for controlling the risk of credit default losses

Scenario analysis deals with quantifying and controlling the risk of extreme losses. Losses up to a certain confidence level, such as the 99th percentile level, are controlled by the use of adequate pricing, provisioning and economic capital. Provisioning for credit risk is discussed in detail in Section 5.2.

Applications of CreditRisk+

5

5.1 Introduction

CreditRisk+ includes several applications of the credit risk modelling technology in the areas of provisioning, setting risk-based credit limits, and portfolio management.

5.2 Provisioning for Credit Risk

Applications
Provisioning
Limits
Portfolio Management

One application of CreditRisk+ is in defining an appropriate credit risk provisioning methodology that reflects the credit losses of the portfolio over several years and hence that more accurately presents the true earnings of the business by matching income with losses.

5.2.1 The Need for Credit Provisions

Generally, current accounting and provisioning policies recognise credit income and credit losses at different times, even though the two events are related. Usually, credit loss provisions are made only when exposures have been identified as non-performing. These provisions are often supplemented with other specific and general credit provisions.

■ The CreditRisk+ credit risk provisioning methodology more accurately reflects the true earnings of the business by matching income with losses.

In relation to any portfolio of credit exposures, there is a statistical likelihood that credit default losses will occur, even though the obligors are currently performing and it is not possible to identify specifically which obligors will default. The level of expected loss reflects the continuing credit risk associated with the firm's existing performing portfolio and is one of the costs of doing credit-related business. This level of expected loss should be taken account of when executing any business that has a credit risk impact.

When default losses are modelled, it can be observed that the most frequent loss amount will be much lower than the average, because, occasionally, extremely large losses are suffered, which have the effect of increasing the average loss. Therefore, a credit provision is required as a means of protecting against distributing excess profits earned during the below average loss years.

5.2.2 Annual Credit Provision (ACP)

The starting point for provisioning is to separate the existing portfolio into a non-performing and a performing portfolio. The non-performing portfolio should be fully provisioned to the expected recovery level available through foreclosure, administration or liquidation. Once fully provisioned, the non-performing portfolio should then be separated out and passed to a specialist team for ongoing management.

As for the performing portfolio, since no default has occurred, one needs a forward-looking provisioning methodology. Under CreditRisk+, the Annual Credit Provision ("ACP") represents the future expected credit loss on the performing portfolio, which is calculated as follows:

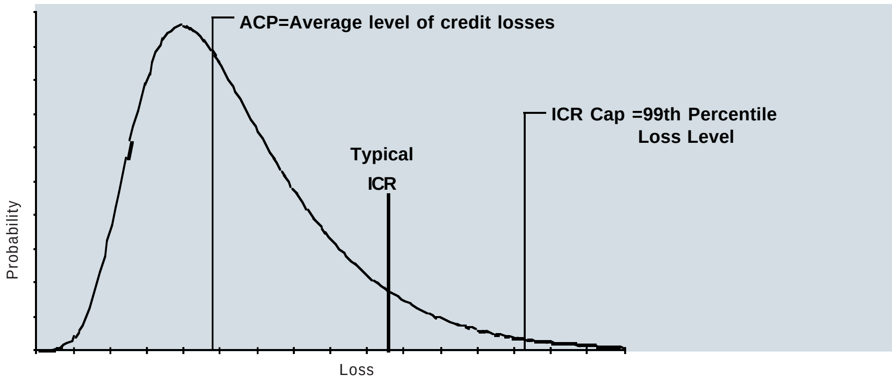
■ The Annual Credit Provision reflects the continuing credit risk associated with the portfolio and is one of the costs of doing business that creates credit risk.

ACP = Exposure x Default Rate x (100% - Recover Rate)

The ACP should be calculated frequently in order to reflect the changing credit quality of the portfolio. The ACP is the first element of the credit provisioning methodology.

5.2.3 Incremental Credit Reserve (ICR)

The ACP represents only the expected or average level of credit losses. As experience shows, actual losses that occur in any one year may be higher or lower than this amount, depending on the economic environment, interest rates, etc. In fact, a better way of viewing the annual credit loss of the portfolio is as a distribution of possible losses (outcomes), whose average equals the ACP but has a small probability of much larger losses. In order to absorb these variations in credit losses from year to year, a second element of the provisioning methodology, the Incremental Credit Reserve ("ICR"), can be established.



◀ Figure 11:
Credit risk provisioning

■ The Incremental Credit Reserve protects against unexpected credit losses and is used to absorb losses that are higher than the expected level.

The CreditRisk+ Model provides information on the distribution of possible losses in the performing credit portfolio. The ICR provides protection against unexpected credit losses (i.e. in excess of the ACP) and is subject to a cap derived from the CreditRisk+ Model (the "ICR Cap"). The ICR Cap represents an extreme case of possible credit losses (e.g. the 99th percentile loss level) on the performing portfolio.

5.2.4 Provisioning for Different Business Lines

The credit risk provisioning methodology described above relates to credit risk arising from a loans business where the income is accounted for on an accruals basis rather than by marking-to-market.

A credit risk provision can also be established for other credit business lines, such as traded bond portfolios and derivatives portfolios. In each case, the CreditRisk+ Model provides the information required in order to establish the provision that ensures that the accounting principle of matching income with losses is maintained. For example, for a portfolio of bonds, part of the expected loss is incorporated within the market price and hence only the incremental credit reserve is required. This is described in the following table.

Table 6: Provisioning for different business lines

Portfolio type	Accounting treatment	Provision
Loan (Counterparty risk)	Accrual (credit neutral)	• ACP (1 year) charge to P&L each year • ICR (1 year)
Derivatives (Counterparty risk)	Mark-to-market (credit neutral)	• ACP (full maturity) held as mark-to-market adjustment • ICR (1 year)
Bond (Issuer risk)	Mark-to-market (credit inclusive)	• ICR (1 year) to support business and protect against distribution of profits

5.2.5 Managing the Credit Risk Provision

As credit defaults occur, loans or exposures are moved from the performing to the non-performing portfolio and hence provisioned to the expected recovery level. This increase in provision is then charged first against the ACP and then, to the extent necessary, against the ICR. To the extent that actual credit losses are less than the ACP within any given year, the balance is credited to the ICR up to the ICR Cap, beyond which the balance is taken into P&L. This ensures that the ICR is replenished during low loss years following a large unexpected loss, but that the ICR never exceeds the ICR Cap.

A worked example can be seen in the table below:

Table 7: Example of credit risk provisioning

Year	1	2	3	4	5
Assumptions					
Actual loan losses	500	600	300	300	650
ACP	500	525	550	610	625
ICR - Initial level	1,900	-	-	-	-
ICR Cap	2,000	2,100	2,200	2,250	2,300
Income Statement					
Operating profit	2,100	2,100	2,205	2,315	2,430
Less: ACP	(500)	(525)	(550)	(610)	(625)
Add: excess unutilised provision over ICR Cap	0	0	0	135	0
Pre-tax profit	1,600	1,575	1,655	1,840	1,805
ICR (pre cap)	1,900	1,825	2,075	2,385	2,225
ICR Cap (as above)	2,000	2,100	2,200	2,250	2,300
Excess unutilised provision over ICR Cap	0	0	0	135	0
ICR (with cap applied)	1,900	1,825	2,075	2,250	2,225

5.3 Risk-Based Credit Limits

A system of individual credit limits is a well-established means of managing credit risk. Monitoring exposures against limits provides a trigger mechanism for identifying potentially unwanted exposures that require active management.

5.3.1 Standard Credit Limits

The system of credit limits may be viewed from a different perspective, if applying the methodologies described within this document.

In particular, in order to equalise a firm's risk appetite between obligors as a means of diversifying its portfolio, a credit limit system could aim to have a large number of exposures with equal expected losses. The expected loss for each obligor can be calculated as the default rate multiplied by the exposure amount less the expected recovery. This means that individual credit limits should be set at levels that are inversely proportional to the default rate corresponding to the obligor rating.

As might be expected, this methodology gives larger limits for better ratings and shorter maturities, but has the benefit of allowing a firm to relate the size and tenor of limits for different rating categories to each other.

■ A system of standard credit limits can be supplemented with portfolio level risk information to manage credit risk.

This approach can be extended to base limits on equalising the portfolio risk contribution for each obligor. A discussion on risk contributions and their use in portfolio management is provided later in this section.

5.3.2 Concentration Limits

Any excess country or industry sector concentration can have a negative effect on portfolio diversification and increase the riskiness of the portfolio. As a result, a comprehensive set of country and industry sector limits is required to address concentration issues in the portfolio. Concentration limits have the effect of limiting the loss from identified scenarios and is a powerful technique for managing "tail" risk and controlling catastrophic losses.

5.4 Portfolio Management

The CreditRisk+ Model makes the process of controlling and managing credit risk more objective by incorporating into a single measure all of the factors that determine the amount of risk.

5.4.1 Introduction

Currently, the primary technique for controlling credit risk is through the use of limit systems, including:

- Individual obligor limits to control the size of exposure
- Tenor limits to control the maximum maturity of transactions with obligors
- Rating exposure limits to control the amount of exposure to obligors of certain credit ratings and
- Concentration limits to control concentrations within countries and industry sectors.

5.4.2 Identifying Risky Exposures

For managing credit risks on a portfolio basis, with the aim of creating a diversified portfolio, a different measure that incorporates the magnitude of the exposure, maturity, credit quality and systematic risk into a single measure is required.

The risk of a particular exposure is determined by four factors: (i) the size of exposure, (ii) the maturity of the exposure, (iii) the probability of default, and (iv) the systematic or concentration risk of the obligor. Credit limits aim to control risk arising from each of these factors individually. However, for managing risks on a portfolio basis, with the aim of creating a diversified portfolio, a different measurement, which incorporates size, maturity, credit quality and systematic risk into a single measure, is required.

5.4.3 Measuring Diversification

The loss distribution and the level of economic capital required to support a portfolio are measures of portfolio diversification that take account of the size, maturity, credit quality and systematic risk of each exposure.

If the portfolio were less diversified, the spread of the distribution curve would be wider and a higher level of economic capital would be required. Conversely, if the portfolio were more diversified, a lower level of economic capital would be required. These measures can be used in managing a portfolio of exposures.

5.4.4 Portfolio Management using Risk Contributions

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The risk contribution of an exposure is defined as the incremental effect on a chosen percentile level of the loss distribution when the exposure is removed from the existing portfolio. If the percentile level chosen is the same as that used for calculating economic capital, the risk contribution is the incremental effect on the amount of economic capital required to support the portfolio.

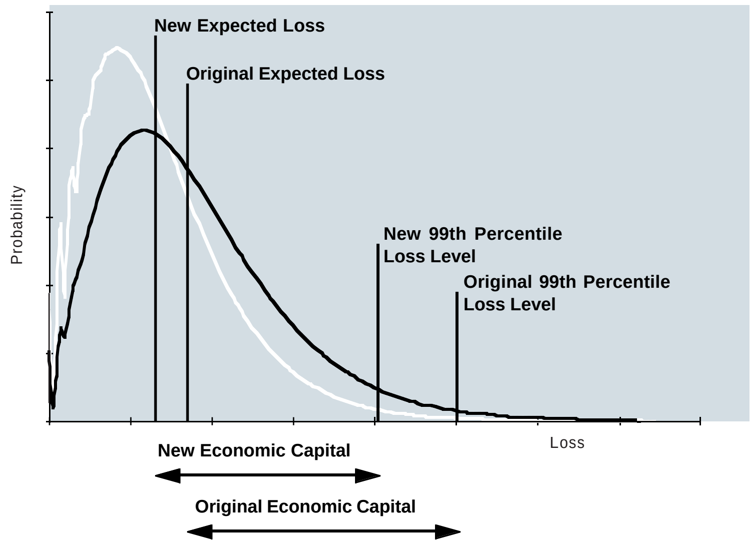
Risk contributions have several features including the following:

In general, a portfolio can be effectively managed by focusing on a relatively few obligors that represent a significant proportion of the risk but constitute a relatively small proportion of the absolute portfolio exposures.

- The total of the risk contributions for the individual obligors is approximately equal to the risk of the entire portfolio.
- Risk contributions allow the effect of a potential change in the portfolio (e.g. the removal of an exposure) to be measured.
- In general, a portfolio can be effectively managed by focusing on a relatively few obligors that represent a significant proportion of the risk but constitute a relatively small proportion of the absolute portfolio exposures.

Therefore, risk contributions can be used in portfolio management. By ranking obligors in decreasing order of risk contribution, the obligors that require the most economic capital can easily be identified.

This is illustrated in the following example. A portfolio was created from which a small number of exposures with the highest risk contributions were removed. The effect on the loss distribution and the levels for the expected loss and the economic capital can be seen in the figure opposite.



◀ Figure 12:
Using risk contributions

■ Information about risk contributions can be used to facilitate risk management and efficient use of economic capital.

The reduction in the 99th percentile loss level is larger than the reduction in the expected loss level, which leads to an overall reduction in the economic capital required to support the portfolio.

5.4.5 Techniques for Distributing Credit Risk

Once obligors representing a significant proportion of the risk have been identified, there are several techniques for distributing credit risk that can be applied. These include the following:

- Collateralisation: In the context of the CreditRisk+ Model, taking collateral has the effect of reducing the severity of the loss given that the obligor has defaulted.
- Asset securitisations: Asset securitisations involve the packaging of assets into a bond, which is then sold to investors.
- Credit derivatives: Credit derivatives are a means of transferring credit risk from one obligor to another, while allowing client relationships to be maintained.

Appendix A - The CreditRisk+ Model

A

A1 Overview of this Appendix

This appendix presents an analytic technique for generating the full distribution of losses from a portfolio of credit exposures. The technique is valid for any portfolio where the default rate for each obligor is small and generates both one-year and multi-year loss distributions.

The appendix applies the concepts discussed in Sections 2 and 3 of this document. The key concepts are:

- Default rates are stochastic.
- The level of default rates affects the incidence of default events but there is no causal relationship between the events.

In order to facilitate the explanation of the CreditRisk+ Model, we first consider the case in which the mean default rate for each obligor in the portfolio is fixed. We then generalise the technique to the case in which the mean default rate is stochastic. The modelling stages of the CreditRisk+ Model and the relationships between the different sections of this appendix are shown in the figure opposite.

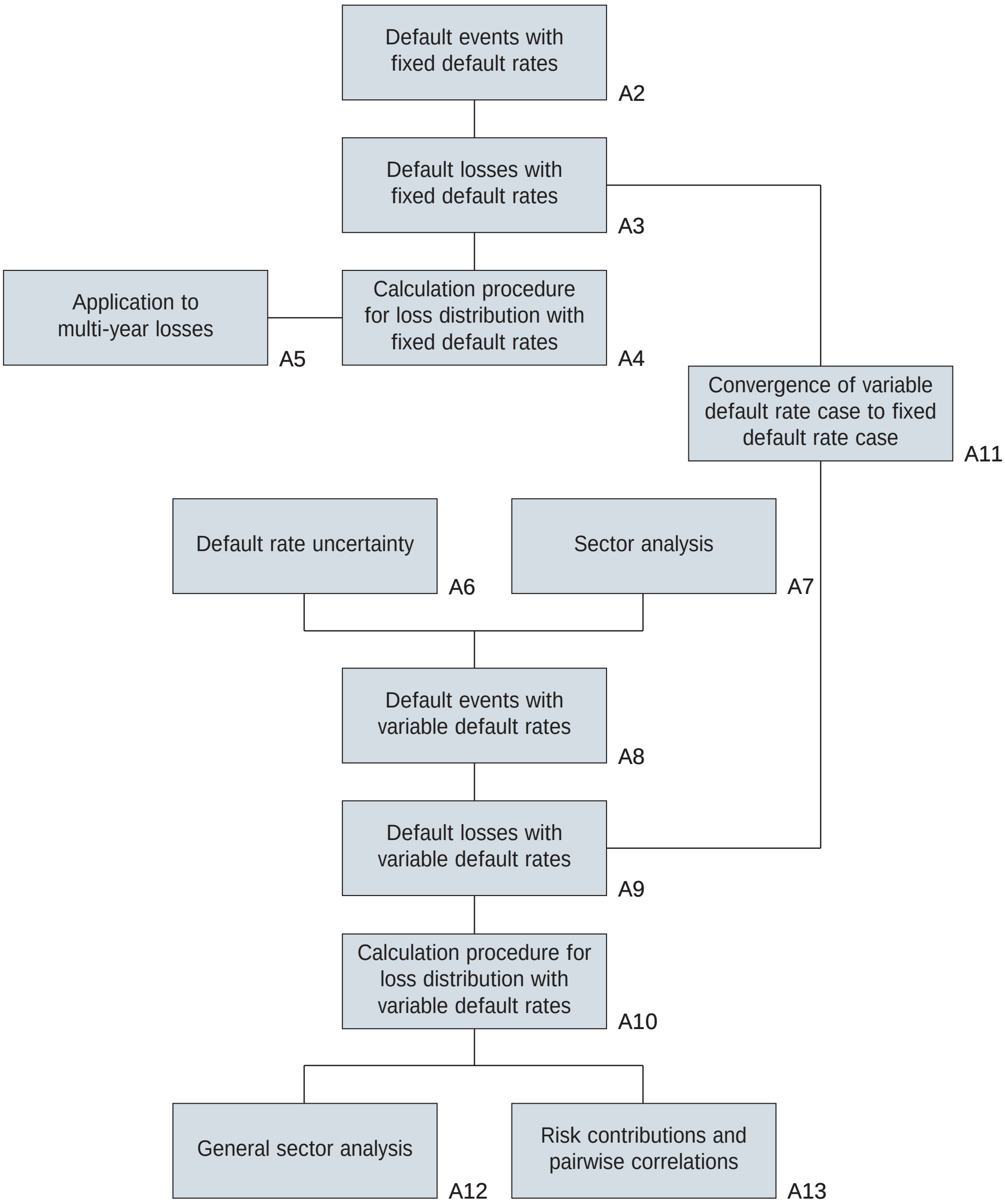


Figure 13:
Flowchart description of Appendix A

A2 Default Events with Fixed Default Rates

In Sections A2 to A5 we develop the theory of the distribution of credit default losses under the assumption that the default rate is fixed for each obligor. Given this assumption and the fact that there is no causal relationship between default events, we interpret default events to be independent. In Sections A6 onwards, the assumption of fixed default rates is relaxed, which introduces dependence between default events. In Section A13 this dependence is quantified by calculating the correlation between default events implied by the CreditRisk+ Model.

A2.1 Default Events

Credit defaults occur as a sequence of events in such a way that it is not possible to forecast the exact time of occurrence of any one default or the exact total number of defaults. In this section we derive the basic statistical theory of such processes in the context of credit default risk.

Consider a portfolio consisting of N obligors. In line with the above assumptions, it is assumed that each exposure has a definite known probability of defaulting over a one-year time horizon. Thus

$$p_A = \text{Annual probability of default for obligor } A \tag{1}$$

To analyse the distribution of losses arising from the whole portfolio, we introduce the probability generating function defined in terms of an auxiliary variable z by

$$F(z) = \sum_{n=0}^{\infty} p(n \text{ defaults}) z^n \tag{2}$$

An individual obligor either defaults or does not default. The probability generating function for a single obligor is easy to compute explicitly as

$$F_A(z) = 1 - p_A + p_A z = 1 + p_A(z-1) \tag{3}$$

As a consequence of independence between default events, the probability generating function for the whole portfolio is the product of the individual probability generating functions. Therefore

$$F(z) = \prod_A F_A(z) = \prod_A (1 + p_A(z-1)) \tag{4}$$

It is convenient to write this in the form

$$\log F(z) = \sum_A \log(1 + p_A(z-1)) \tag{5}$$

Suppose next that the individual probabilities of default are uniformly small. This is characteristic of portfolios of credit exposures. Given that the probabilities of default are small, powers of those probabilities can be ignored. Thus, the logarithms can be replaced using the expression[4]

$$\log(1 + p_A(z-1)) = p_A(z-1) \tag{6}$$

and, in the limit, equation (5) becomes

$$F(z) = e^{\sum_A p_A(z-1)} = e^{\mu(z-1)} \tag{7}$$

where we write

$$\mu = \sum_A p_A \tag{8}$$

for the expected number of default events in one year from the whole portfolio.

To identify the distribution corresponding to this probability generating function, we expand F(z) in its Taylor series:

$$F(z) = e^{\mu(z-1)} = e^{-\mu} e^{\mu z} = \sum_{n=0}^{\infty} \frac{e^{-\mu} \mu^n}{n!} z^n \tag{9}$$

[4] This approximation ignores terms of degree 2 and higher in the default probabilities. The expressions derived from this approximation are exact in the limit as the probabilities of default tend to zero, and give good approximations in practice.

Thus if the probabilities of individual default are small, although not necessarily equal, then from equation (9) we deduce that the probability of realising *n* default events in the portfolio in one year is given by

$$\text{Probability (n defaults)} = \frac{e^{-\mu}\mu^{n}}{n!} \qquad (10)$$

A2.2 Summary

In equation (10) we have obtained the well-known Poisson distribution for the distribution of the number of defaults under our initial assumptions. The following should be noted:

- The distribution has only one parameter, the expected number of defaults μ. The distribution does not depend on the number of exposures in the portfolio or the individual probabilities of default provided that they are uniformly small.
- There is no necessity for the exposures to have equal probabilities of default; indeed, the probability of default can be individually specified for each exposure if sufficient information is available.

The Poisson distribution with mean μ can be shown to have standard deviation given by $\sqrt{\mu}$. Historical evidence of the standard deviation of default event frequencies exists in the form of year-on-year default rate tables. Such data suggests that the actual standard deviation is invariably much larger than $\sqrt{\mu}$. Thus, our initial assumption of fixed default rates cannot account for observed data. Before addressing this in Section A6, we first consider the derivation of the credit loss distribution from the results on default events above, retaining our initial assumptions for now.

A3 Default Losses with Fixed Default Rates

A3.1 Introduction

Under our initial assumptions, the distribution of numbers of defaults in a portfolio of exposures in one year has been obtained. However, our main objective is to understand the likelihood of suffering given levels of loss from the portfolio, rather than given numbers of defaults. The distributions are different because the same level of default loss could arise equally from a single large default or from a number of smaller defaults in the same year. Unlike the variation of default probability between exposures, which does not influence the distribution of the total number of defaults, differing exposure amounts result in a loss distribution that is not Poisson in general. Moreover, information about the distribution of different exposures is essential to the overall distribution. However, it is possible to describe the overall distribution because its probability generating function has a simple closed form amenable to computation.

A3.2 Using Exposure Bands

The first step in obtaining the distribution of losses from the portfolio in an amenable form is to group the exposures in the portfolio into bands. This has the effect of significantly reducing the amount of data that must be incorporated into the calculation.

Banding introduces an approximation into the calculation. However, provided the number of exposures is large and the width of the bands is small compared with the average exposure size characteristic of the portfolio, the approximation is insignificant. Intuitively, this corresponds to the fact that the precise amounts of exposures in a portfolio cannot be critical in determining the overall risk.

Once the appropriate notation has been set up, an estimate of the effect of banding on the mean and standard deviation of the portfolio is given below.

A3.3 Notation

In this section, the notation used for the exposure banding described above is detailed.

Reference	Symbol
Obligor	A
Exposure	L_A
Probability of default	P_A
Expected Loss	λ_A

In order to perform the calculations, a unit amount of exposure *L*, denominated in a base currency, is chosen. For each obligor *A*, define numbers ε_A and ν_A by writing

$$L_A = L \times \nu_A \text{ and } \lambda_A = L \times \varepsilon_A \tag{11}$$

Thus, ν_A and ε_A are the exposure and expected loss, respectively, of the obligor, expressed as multiples of the unit.

The key step is to round each exposure size ν_A to the nearest whole number. This step replaces each exposure amount L_A by the nearest integer multiple of *L*. If a suitable size for the unit *L* is chosen, then, after the rounding has been performed for a large portfolio, there will be a relatively small number of possible values for ν_A each shared by several obligors.

The portfolio can then be divided into *m* exposure bands, indexed by *j*, where $1 \leq j \leq m$. With respect to the exposure bands, we make the following definitions

Reference	Symbol
Common exposure in Exposure Band j in units of L	ν_j
Expected loss in Exposure Band j in units of L	ε_j
Expected number of defaults in Exposure Band j	μ_j

The following relations hold, expressing the expected loss in terms of the probability of default events

$$\varepsilon_j = \nu_j \times \mu_j \text{ ;hence } \mu_j = \frac{\varepsilon_j}{\nu_j} = \sum_{A:\nu_A = \nu_j} \frac{\varepsilon_A}{\nu_A} \tag{12}$$

Note that, because we have rounded the ν_j to make them whole numbers, the expected loss ε_A will be affected, by equation (12) unless a compensating rounding adjustment is made to the expected number of default events μ_j. If no adjustment is made, the rounding process will result in a small rounding up of the expected loss. Under the assumption stated above, that the unit size is small relative to the typical exposure size of the portfolio, these approaches each have an immaterial effect on the loss distribution. In the rest of this Appendix it is assumed that an adjustment to the default probabilities to preserve the expected losses is made. Provided the exposure sizes are rounded up, then, as will be shown in Section A4.2, the rounding leads to a small overstatement of the standard deviation.

As in equation (8), let μ stand for the total expected number of default events in the portfolio in one year. Since μ is the sum of the expected number of default events in each exposure band, we have

$$\mu = \sum_{j=1}^{m} \mu_j = \sum_{j=1}^{m} \frac{\varepsilon_j}{v_j} \tag{13}$$

A3.4 The Distribution of Default Losses

We have analysed the distribution of default events under our initial assumptions. We now proceed to derive the distribution of default losses.

Intuitively, the default loss analysis involves a second element of randomness, because some defaults lead to larger losses than others through the variation in exposure amounts over the portfolio. As with default events, the second random effect is best described mathematically through its probability generating function. Thus, let G(z) be the probability generating function for losses expressed in multiples of the unit L of exposure:

$$G(z) = \sum_{n=0}^{\infty} p(\text{aggregate losses} = n \times L) z^n \tag{14}$$

The exposures in the portfolio are assumed to be independent. Therefore, the exposure bands are independent, and the probability generating function can be written as a product over the exposure bands

$$G(z) = \prod_{i=1}^{m} G_i(z) \tag{15}$$

However, by treating each exposure band as a portfolio and using equation (9), we obtain

$$G_j(z) = \sum_{n=0}^{\infty} p(n \text{defaults}) z^{n v_j} = \sum_{n=0}^{\infty} \frac{e^{-\mu_j} \mu_j^n}{n!} z^{n v_j} = e^{-\mu_j + \mu_j z^{v_j}} \tag{16}$$

Therefore

$$G(z) = \prod_{j=1}^{m} e^{-\mu_j + \mu_j z^{v_j}} = e^{-\sum_{j=1}^{m} \mu_j + \sum_{j=1}^{m} \mu_j z^{v_j}} \tag{17}$$

This is the desired formula for the probability generating function for losses from the portfolio as a whole. In the next section, we show how to use the probability generating function to derive the actual distribution of losses under our initial assumptions.

For later reference, equation (17) can be restated in a slightly different form. First, define a polynomial P(z) as follows

$$P(z) = \frac{\sum_{j=1}^{m} \mu_j z^{v_j}}{\mu} = \frac{\sum_{j=1}^{m} \left(\frac{\varepsilon_j}{v_j}\right) z^{v_j}}{\sum_{j=1}^{m} \left(\frac{\varepsilon_j}{v_j}\right)} \tag{18}$$

where we have used equations (12) and (13) for the total number μ of defaults in the portfolio. The probability generating function in equation (17) can now be expressed as

$$G(z) = e^{\mu(P(z)-1)} = F(P(z)) \tag{19}$$

This functional form for G(z) expresses mathematically the compounding of two sources of uncertainty arising, respectively, from the Poisson randomness of the incidence of default events and the variability of exposure amounts within the portfolio.

Note that G(z) depends only on the data ν and ε. Therefore, to obtain the distribution of losses for a large portfolio of credit risks, all that is needed is knowledge of the different sizes of exposures ν within the portfolio, together with the share ε of expected loss arising from each exposure size. This is typically a very small amount of data, even for a large portfolio.

A4 Loss Distribution with Fixed Default Rates

A4.1 Calculation Procedure

In this section, a computationally efficient means of deriving the actual distribution of credit losses is derived from the probability generating function given by equation (17). In Section A10, this approach will be generalised to compute the distribution for the CreditRisk+ Model.

For n an integer let A_n be the probability of a loss of nxL, or n units from the portfolio. We wish to compute A_n efficiently. Comparing the definition in equation (14) with the Taylor series expansion for G(z), we have

$$p(\text{loss of nL}) = \frac{1}{n!}\frac{d^n G(z)}{dz^n}\bigg|_{z=0} = A_n \tag{20}$$

In our case G(z) is given in closed form by equation (17). Using Leibnitz's formula we have

$$\begin{aligned}\frac{1}{n!}\frac{d^n G(z)}{dz^n}\bigg|_{z=0} &= \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\left(G(z).\frac{d}{dz}\sum_{j=1}^{m}\mu_j z^{\nu_j}\right)\bigg|_{z=0}\\ &= \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^{n-k-1}}{dz^{n-k-1}}G(z)\frac{d^{k+1}}{dz^{k+1}}\left(\sum_{j=1}^{m}\mu_j z^{\nu_j}\right)\bigg|_{z=0}\end{aligned} \tag{21}$$

However

$$\frac{d^{k+1}}{dz^{k+1}}\left(\sum_{j=1}^{m}\mu_j z^{\nu_j}\right)\bigg|_{z=0} = \begin{cases}\mu_j(k+1)! & \text{if } k=\nu_j-1 \text{ for some } j\\ 0 & \text{otherwise}\end{cases} \tag{22}$$

and by definition

$$\frac{d^{n-k-1}}{dz^{n-k-1}}G(z)\bigg|_{z=0} = (n-k-1)!A_{n-k-1} \tag{23}$$

Therefore

$$A_n = \sum_{\substack{k\le n-1\\ k=\nu_j-1 \text{ for some } j}}\frac{1}{n!}\binom{n-1}{k}(k+1)!(n-k-1)!\mu_j A_{n-k-1} = \sum_{j:\nu_j\le n}\frac{\mu_j\nu_j}{n}A_{n-\nu_j} \tag{24}$$

Using the relation $\varepsilon_j = \nu_j \times \mu_j$ from equation (12), the following recurrence relationship is obtained

$$A_n = \sum_{j:\nu_j\le n}\frac{\varepsilon_j}{n}A_{n-\nu_j} \tag{25}$$

This recurrence relationship allows very quick computation of the distribution. In order to commence the computation, we have the following formula for the first term, which expresses the probability of no loss arising from the portfolio

$$A_0 = G(0) = F(P(0)) = e^{-\mu} = e^{-\sum_{j=1}^{m}\frac{\varepsilon_j}{\nu_j}} \tag{26}$$

Again, it is worthwhile to note that the calculation depends only on knowledge of ε and ν. In practice, these represent a very small amount of data even for a large portfolio consisting of many exposures.

A4.2 Precision Using Exposure Bands

The banding process described in Section A3 introduces a small degree of approximation into the data. In this section, we show that the approximation error is normally not material by considering the effect on the portfolio mean and standard deviation.

In terms of the notation above, the total portfolio expected loss ε and total portfolio standard deviation σ are

$$\varepsilon = \sum_{j=1}^{m} \varepsilon_j \ ; \ \sigma^2 = \sum_{j=1}^{m} \nu_j \times \varepsilon_j \qquad (27)$$

where the expected loss and standard deviation are expressed in the chosen unit *L*.

In order to represent the banding, suppose that the above are expressions for the "true" mean and standard deviation, but that now the ν are rounded to integer multiples of the unit as explained above. This process introduces an error; however, write

$$\hat{\nu}_j = \nu_j + \tau_j \text{ where } 0 \le \tau_j \le 1 \qquad (28)$$

Each τ_j is at most of absolute size one. It is assumed that the exposures are rounded up, so that each τ_j is positive.

The expected loss is unaffected by the method of rounding chosen, because its expression is independent of the banded exposure amounts. This was noted above.

For the standard deviation, we have

$$\sigma^2 \le \hat{\sigma}^2 = \sum_{j=1}^{m} \hat{\nu}_j \times \varepsilon_j = \sigma^2 + \sum_{j=1}^{m} \tau_j \times \varepsilon_j \le \sigma^2 + \sum_{j=1}^{m} \varepsilon_j = \sigma^2 + \varepsilon \qquad (29)$$

where ε is the expected loss for the portfolio. Taking square roots and neglecting higher-order terms in the Taylor series we obtain

$$\sigma \le \hat{\sigma} \le \sigma\left(1 + \frac{\varepsilon}{2\sigma^2}\right) = \sigma + \frac{\varepsilon}{2\sigma} \qquad (30)$$

For a real portfolio, the expected loss ε and the quantity 2σ are of the same order. We conclude that:

- The expected loss calculated by the model is unaffected by the banding process.
- The standard deviation is overstated by an amount comparable with the chosen unit size.

A5 Application to Multi-Year Losses

A5.1 Introduction

The recurrence relation above was derived on the basis of a one-year loss distribution. In this section it is shown how the initial model can be applied over a multi-year time horizon.

As in the discussion over a one-year horizon, consider a portfolio of obligors with small probabilities of default. For simplicity it is assumed that the future of the portfolio is divided into years. The exposures are permitted to vary from year to year. In particular, each exposure has an individual maturity corresponding to the normal maturity of bonds, loans or other instruments.

A5.2 Term Structure of Default

In order to address a multi-year horizon, marginal rates of default must be specified for each future year for each obligor in the portfolio. Collectively, such marginal default rates give the term structure of default for the portfolio.

A5.3 Notation

Fix the following notation:

Reference	Symbol
Probability of default of exposure j in year t	$p_j^{(t)}$
Amount of exposure j in year t	$L_j^{(t)} = L\nu_j^{(t)}$
Expected loss in exposure j in year t	$\lambda_j^{(t)} = L\varepsilon_j^{(t)}$

As for the one-year discussion, L is the unit of exposure and the $\nu_j^{(t)}$ are dimensionless whole numbers. Under the natural assumption that defaults by the same exposure in different years are mutually exclusive, the probability generating function for multi-year losses from a single exposure is given by

$$G_j(z) = 1 - \sum_{t=0}^{T} p_j^{(t)} + \sum_{t=0}^{T} p_j^{(t)} z^{\nu_j^{(t)}} = 1 + \sum_{t=0}^{T} p_j^{(t)} (z^{\nu_j^{(t)}} - 1) \tag{31}$$

For the generating function of total losses, we have

$$\log G(z) = \sum_j \log\left(1 + \sum_{t=1}^{T} p_j^{(t)} (z^{\nu_j^{(t)}} - 1)\right) \tag{32}$$

In the limit of small probabilities of default we argue as for equation (6) to obtain

$$\log\left(1 + \sum_{t=1}^{T} p_j^{(t)} (z^{\nu_j^{(t)}} - 1)\right) = \sum_{t=1}^{T} p_j^{(t)} (z^{\nu_j^{(t)}} - 1) \tag{33}$$

and we obtain

$$\log G(z) = \sum_j \sum_{t=1}^{T} p_j^{(t)} (z^{\nu_j^{(t)}} - 1) = \sum_{\nu} z^{\nu} \left(\sum_{j,t:\nu_j^{(t)}=\nu} p_j^{(t)} \right) \tag{34}$$

and

$$\sum_{j\,t:\nu_j^{(t)}=\nu} p_j^{(t)} = \sum_{t=1}^{T} \sum_{j;\nu_j^{(t)}=\nu} p_j^{(t)} = \sum_{t=1}^{T} \sum_{j,:\nu_j^{(t)}=\nu} \frac{\varepsilon_j^{(t)}}{\nu_j^{(t)}} \tag{35}$$

The probability generating function for the loss distribution is therefore given by

$$G(z) = e^{\sum_{j,t} \frac{\varepsilon_j^{(t)}}{\nu_j^{(t)}} (z^{\nu_j^{(t)}} - 1)} \tag{36}$$

This has the same form as the one year probability generating function (17). Therefore, the recurrence relation given by equation (25) for the distribution of losses over one year is also applicable to the calculation of the multi-year distribution of losses

$$A_n = \sum_{j\,t;\nu_j^{(t)} \le n} \frac{\varepsilon_j^{(t)}}{n} A_{n-\nu_j^{(t)}} \tag{37}$$

A6 Default Rate Uncertainty

The previous sections developed the theory of the loss distribution from a portfolio of obligors, each of which has a fixed probability of default. In the following sections, the CreditRisk+ Model will be developed from this theory by incorporating default rate uncertainty and sector analysis. These concepts are introduced in this section and Section A7 respectively.

Published statistics on the incidence of default events, for example among rated companies in a given country, show that the number of default events, and therefore the average probability of default for such entities, exhibits wide variation from year to year[5].

Such year-on-year statistics may be thought of as samples from a random variable whose expected value represents an average rate of default over many years. The appearance of randomness is due to the incidence of factors, such as the state of the economy, that influence the fortunes of obligors. The standard deviation of the variable measures our uncertainty as to the actual default rate that will be exhibited over a given year. Owing to default rate uncertainty, there is a chance that default rates will turn out to be higher over, for example, the next year than their average over many years suggests. This in turn leads to a higher chance of experiencing extreme losses.

The situation may be summarised by the following three intuitive facts about default rate uncertainty:

- Observed default probabilities are volatile over time, even for obligors having comparable credit quality.
- The variability of default probabilities can be related to underlying variability in a relatively small number of background factors, such as the state of the economy, which affect the fortunes of obligors. For example, a downward trend in the state of the economy may make most obligors in a portfolio more likely to default.
- However, a change in the economy or another factor will not cause obligors to default with certainty. Whatever the state of the economy, actual defaults should still be relatively rare events. Therefore the analysis above which considered rare events is relevant in a suitably modified form.

The second point made above is that uncertainty arises from factors that may affect a large number of obligors in the same way. In order to measure this effect and hence quantify the impact of individual default rate volatilities at the portfolio level, the concept of sector analysis is necessary. This concept is introduced in the next section.

A7 Sector Analysis

A7.1 Introduction

It was noted above that the variability of default rates can be related to the influence of a relatively small number of background factors on the obligors within a portfolio. In order to measure the effect of these factors, it is necessary to quantify the extent to which each factor has an influence on a given portfolio of obligors. A factor such as the economy of a particular country may be considered to have a uniform influence on obligors whose domicile is within that country, but relatively little influence on other obligors in a multinational portfolio.

In this section, the measurement of background factors is addressed by dividing the obligors among different sectors, where each sector is a collection of obligors under the common influence of a major factor affecting default rates. An initial example might be a division of the portfolio according to the country of domicile of each obligor. In Section A12, a more general sector analysis, which allows for the fact that, in reality, obligors may be under the simultaneous influence of a number of major factors, is presented.

[5] If the default rates of obligors were fixed, default events would still have a non-zero standard deviation arising from the randomness of the default events themselves. However, as remarked in Section A2.2, comparison with historic data shows that observed volatility is far higher than can be accounted for in this way.

A7.2 Further Notation

New notation is needed to keep track of the division of the portfolio into sectors and to record the volatility of the default rate for each sector. Write S_k: $1 \le k \le n$ for the sectors, each of which should be thought of for now as a subset of the collection of obligors.

The CreditRisk+ Model regards each sector as driven by a single underlying factor, which explains the variability over time in the average total default rate measured for that sector. The underlying factor influences the sector through the total expected rate of defaults in the sector, which is then modelled as a random variable with mean μ_k and standard deviation σ_k specified for each sector. The standard deviation will reflect the degree to which the probabilities of default of the obligors in the portfolio are liable to all be more or less than their average levels. For example, in a sector consisting of a large number of obligors of low credit quality, the mean default rate might be 5% per annum and the standard deviation of the actual default rate might be a similar quantity. Then there will be a substantial chance in any year of the actual average probability of default in the sector being, say, 10% instead of 5%. In turn it is much more likely that, say, 12% of the obligors will actually default in that year. Had the standard deviation been zero, reflecting that we were certain about the probability of each obligor defaulting, then a year in which as many as 12% of the obligors default would have been a much more remote possibility.

The table below summarises the new notation to specify the sector decomposition of the portfolio. In particular, for each sector we introduce a random variable x_k representing the average default rate over the sector. The mean of x_k is μ_k and the standard deviation is σ_k.

Sector	$S_k : 1 \le k \le n$
Random variable representing the mean number of defaults	x_k
Long-term annual average number of defaults - mean of x_k	μ_k
Standard deviation of x_k	σ_k

For each sector, the data requirements are set out below. Our original notation set up in Section A3.3 is also repeated for comparison

Exposure Data within Sector	Previous Notation	New Notation
Base unit of exposure	L	L
Exposure sizes in units	$L_j = L\nu_j$ $1 \le j \le m$	$L_j^{(k)} = L\nu_j^{(k)}$ $1 \le k \le n; 1 \le j \le m(k)$
Expected loss in each exposure band in units	$\lambda_j = L\varepsilon_j$ $1 \le j \le m$	$\lambda_j^{(k)} = L\varepsilon_j^{(k)}$ $1 \le k \le n; 1 \le j \le m(k)$

The mean μ_k is related to the expected loss data by the following relation which is the analogue of equation (13):

$$\mu_k = \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} \tag{38}$$

A7.3 Estimating the Variability of the Default Rate

For each sector, in addition to the expected total rate of default μ_k over the sector given by equation (38), we must specify a standard deviation σ_k of the total expected rate of default. We discuss a convenient way to estimate the standard deviation by reference to equation (38) for the mean. Although equation (38) is expressed in terms of exposure bands, it can equivalently be expanded as a sum over all the obligors in the sector

$$\mu_k = \sum_A \frac{\varepsilon_A}{v_A} \tag{39}$$

where the summation extends over all obligors A belonging to sector *k*, and the relation

$$\frac{\varepsilon_A}{v_A} = p_A \tag{40}$$

expresses the average probability of default of the obligor over the time period. To obtain an estimate of the standard deviation of each sector, we assume that, together with a probability of default p_A, a standard deviation σ_A has been assigned for the default rate for each obligor within the sector. A convenient way to do this is to assume that the standard deviation depends on the credit quality of the obligor. This pragmatic method assumes that the credit quality of the obligors within a sector is a more significant influence on the volatility of the expected default frequency than the nature of the sector.

We obtain an estimate of σ_k from the set σ_A of obligor standard deviations by an averaging process. Recall that only one random variable, x_k is held to account for the uniform variability of each of the probabilities of default. That is, the actual default probability for each obligor in the sector will be modelled as a random variable proportional to x_k, whose mean is equal to the specified mean default rate for that obligor. To express this dependence write x_A for the random default probability of the single obligor A. Our assumption can then be written

$$x_A = \frac{\varepsilon_A}{v_A}\frac{x_k}{\mu_k} \tag{41}$$

Note that the mean of x_A is correctly specified as p_A by this equation. Assuming equation (41), in particular, we have

$$\sum_A \sigma_A = \sum_A \frac{\varepsilon_A}{v_A}\frac{\sigma_k}{\mu_k} = \sigma_k \frac{1}{\mu_k}\sum_A \frac{\varepsilon_A}{v_A} = \sigma_k \tag{42}$$

where we have used equation (39). The sum runs over all obligors in the sector. We estimate the standard deviation of this sector so as to ensure that this condition holds. Thus the standard deviation of the mean default rate for a sector is estimated as the sum of the estimated standard deviations for each obligor in the sector. An alternative and more intuitive description of the standard deviation σ_k determined in this way is that the ratio of σ_k to the mean μ_k is an average of the ratio of standard deviation to mean for each obligor, weighted by their contribution to the default rate. This is easily seen as follows. By equation (42)

$$\frac{\sigma_k}{\mu_k} = \frac{\sum_A \sigma_A}{\sum_A p_A} = \frac{\sum_A p_A \left(\frac{\sigma_A}{p_A}\right)}{\sum_A p_A} \tag{43}$$

According to historical experience, the ratio σ_A / p_A is typically of the order of one, so that the standard deviation of the number of defaults observed year on year among obligors of similar credit quality is typically of the same order as the average annual number of defaults. Equation (43) shows that the same is true for each sector, as one would expect. In the absence of detailed data, the obligor specific estimates of the ratio σ_A / p_A can be replaced by a single flat ratio. Then, writing ω_k for this uniform ratio, equation (43) reduces to the simple form

$$\sigma_k = \omega_k \times \mu_k \tag{44}$$

If the nature of the sector made it more appropriate to estimate the standard deviation σ_k directly, this would be equivalent to estimating the flat ratio ω_k directly.

A8 Default Events with Variable Default Rates

A8.1 Conditional Default Rate

In this section, the distribution of default events for the CreditRisk+ Model is obtained. This is achieved by calculation of the probability generating function. Most of the work has been done already in the calculation of the probability generating function in equation (7) when the default rate is fixed. As in equation (2), the probability generating function for default events is written

$$F(z) = \sum_{n=0}^{\infty} p(\text{n defaults}) z^n \tag{45}$$

Because the sectors are independent, F(z) can be written as a product over the sectors

$$F(z) = \prod_{k=1}^{n} F_k(z) \tag{46}$$

We therefore focus on the determination of F(z) for a single sector. In the notation of Section A7, the average default rate in sector *k* is a random variable, written x_k, with mean μ_k and standard deviation σ_k. Conditional on the value of x_k, we can write down the probability generating function for the distribution of default events as follows

$$F_k(z) \big| [x_k = x] = e^{x(z-1)} \tag{47}$$

where equation (7) has been used. Suppose that x_k has probability density function $f_k(x)$, so that

$$P(x \le x_k \le x + dx) = f_k(x)dx \tag{48}$$

Then, the probability generating function for default events in one sector is the average of the conditional probability generating function given by equation (47) over all possible values of the mean default rate, as the following computation shows:

$$F_k(z) = \sum_{n=0}^{\infty} P(\text{n defaults}) z^n = \sum_{n=0}^{\infty} z^n \int_{x=0}^{\infty} P(\text{n defaults} \,|\, x) f(x)dx = \int_{x=0}^{\infty} e^{x(z-1)} f(x)dx \tag{49}$$

In order to obtain an explicit formula for the probability generating function, an appropriate distribution for X_k must be chosen. We make the key assumption that x_k has the Gamma distribution with mean μ_k and standard deviation σ_k.

The Gamma distribution is chosen as an analytically tractable two-parameter distribution. Before proceeding to evaluate equation (49) explicitly, the basic properties of the Gamma distribution are stated.

A8.2 Properties of the Gamma Distribution

The Gamma distribution, written $\Gamma(\alpha, \beta)$, is a skew distribution, which approximates to the Normal distribution when its mean is large. The probability density function for a $\Gamma(\alpha, \beta)$ - distributed random variable X is given by

$$P(x \le X \le x + dx) = f(x)dx = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha-1} dx \qquad (50)$$

where $\Gamma(\alpha) = \int_{x=0}^{\infty} e^{-x} x^{\alpha-1} dx$ is the Gamma function.

The Gamma distribution $\Gamma(\alpha, \beta)$ is a two parameter distribution, fully described by its mean and standard deviation. These are related to the defining parameters as follows

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2 \qquad (51)$$

Hence, for sector k, the parameters of the related Gamma distribution are given by

$$\alpha_k = \mu_k^2 / \sigma_k^2 \text{ and } \beta_k = \sigma_k^2 / \mu_k \qquad (52)$$

A8.3 Distribution of Default Events in a Single Sector

With the choice of Gamma distribution for the function $f(x)$, the expression for the probability generating function

$$F_k(z) = \int_{x=0}^{\infty} e^{x(z-1)} f(x) dx \qquad (53)$$

given by equation (49), can be directly evaluated. By substitution, change of variable and definition of the Gamma integral

$$F_k(z) = \int_{x=0}^{\infty} e^{x(z-1)} \frac{e^{-\frac{x}{\beta}} x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} dx = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{y=0}^{\infty} \left(\frac{y}{\beta^{-1}+1-z}\right)^{\alpha-1} e^{-y} \frac{dy}{\beta^{-1}+1-z} \qquad (54)$$

$$= \frac{\Gamma(\alpha)}{\beta^{\alpha}\Gamma(\alpha)(1+\beta^{-1}-z)^{\alpha}} = \frac{1}{\beta^{\alpha}(1+\beta^{-1}-z)^{\alpha}}$$

Upon rearrangement this becomes, for sector k

$$F_k(z) = \left(\frac{1-p_k}{1-p_k z}\right)^{\alpha_k} \text{ where } p_k = \frac{\beta_k}{1+\beta_k} \qquad (55)$$

This is the probability generating function of the distribution of default events arising from sector k.

It is possible to identify the distribution of default events underlying this probability generating function. By expanding $F_k(z)$ in its Taylor series

$$F_k(z) = (1-p_k)^{\alpha_k} \sum_{n=1}^{\infty} \binom{n+\alpha_k-1}{n} p_k^n z^n \qquad (56)$$

the following explicit formula is obtained

$$P(\text{n defaults}) = (1-p_k)^{\alpha_k} \binom{n+\alpha_k-1}{n} p_k^n \qquad (57)$$

This can be identified as the probability density of the Negative Binomial distribution.

A8.4 Summary

The portfolio has been divided into *n* sectors with annual default rates distributed according to

$$\Gamma(\alpha_k, \beta_k) \tag{58}$$

The probability generating function for default events from the whole portfolio is given by

$$F(z) = \prod_{k=1}^{n} F_k(z) = \prod_{k=1}^{n} \left(\frac{1-p_k}{1-p_k z} \right)^{\alpha_k} \tag{59}$$

where the parameters α_k, β_k and p_k are given by

$$\alpha_k = \mu_k^2 / \sigma_k^2 \;\; ; \;\; \beta_k = \sigma_k^2 / \mu_k \text{ and } p_k = \beta_k / (1+\beta_k) \tag{60}$$

The default event distribution for each sector is Negative Binomial. The default event distribution for the whole portfolio is not Negative Binomial in general but is an independent sum of the Negative Binomial sector distributions. The corresponding product decomposition of the probability generating function is given by equation (59).

A9 Default Losses with Variable Default Rates

A9.1 Introduction

The probability generating function in equation (59) gives full information about the occurrence of default events in the portfolio. In order to pass from default events to default losses, this distribution must be compounded with the information about the distribution of exposures. In Section A3.4, we performed this process conditional on a fixed mean default rate. We now generalise this process to incorporate the volatility of default rates.

A9.2 The Distribution of Default Losses

By analogy with equation (14), we introduce a second probability generating function *G(z)*, the probability generating function for losses from the portfolio. Thus let

$$G(z) = \sum_{n=0}^{\infty} p(\text{aggregate losses} = n \times \text{L}) z^n \tag{61}$$

be the probability generating function of the distribution of loss amounts. We seek a closed form expression for G(z) and a means of efficiently computing G(z).

As for the distribution of default events, sector independence gives a product decomposition of the probability generating function

$$G(z) = \prod_{k=1}^{n} G_k(z) \tag{62}$$

where $G_k(z)$ is the loss probability generating function for sector *k*, $1 \le k \le n$.

By analogy with equation (18), we define polynomials $P_k(z)$, $1 \le k \le n$, by

$$P_k(z) = \frac{\sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} \right) z^{\nu_j^{(k)}}}{\sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} \right)} = \frac{1}{\mu_k} \sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} \right) z^{\nu_j^{(k)}} \tag{63}$$

where the expression for μ_k in equation (38) has been used. The $P_k(z)$ provide the link between default events and losses, because the following relation holds

$$G_k(z) = F_k(P_k(z)) \tag{64}$$

This is directly analogous to the formula *G(z)=F(P(z))* obtained in equation (19) for a fixed mean default rate, except that there is now one such relation for each sector. In order to see that the relation continues to hold in the present case, we expand equation (63) as a sum over individual obligors belonging to sector *k*. Thus

$$P_k(z) = \frac{\sum_A \frac{\varepsilon_A}{\nu_A} z^{\nu_A}}{\sum_A \frac{\varepsilon_A}{\nu_A}} = \frac{1}{\mu_k} \sum_A \frac{\varepsilon_A}{\nu_A} z^{\nu_A} \tag{65}$$

By equation (41) we have

$$e^{-\sum_A x_A + \sum_A x_A z^{\nu_A}} = e^{\sum_A x_A \left(z^{\nu_A} - 1\right)} = e^{\frac{x_k}{\mu_k} \sum_A \frac{\varepsilon_A}{\nu_A}\left(z^{\nu_A} - 1\right)} = e^{x_k (P_k(z) - 1)} \tag{66}$$

The left hand side of equation (66) is the probability generating function of the distribution of losses where each obligor A has default rate x_A. This can be seen by comparing equation (17) - the expressions are the same, except that in equation (17) terms with the same exposure amount have been collected.

Just as in equation (53), which expresses $F_k(z)$ as an integral of the Poisson probability generating function over the space of possible values of the random variable x_k, a conditional probability argument shows that $G_k(z)$ is the integral of the left hand side of equation (66) over the same space. Thus

$$G_k(z) = \sum_{n=0}^{\infty} z^n \int_{x_k=0}^{\infty} P(\text{Loss of nL} \,|\, x_k) f_k(x_k) dx_k = \int_{x_k=0}^{\infty} e^{\sum_A x_A (z^{\nu_A} - 1)} f_k(x_k) dx_k \tag{67}$$

$$= \int_{x_k=0}^{\infty} e^{x_k (P_k(z) - 1)} f_k(x_k) dx_k$$

Where the last step follows from equation (66). By substitution into equation (55) and taking the product over each sector, we obtain

$$G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n} \left(\frac{1 - p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}} \right)^{\alpha_k} \tag{68}$$

This is a closed form expression for the probability generating function. In the next section a recurrence relation for computing the distribution of losses from this expression is derived.

A10 Loss Distribution with Variable Default Rates

In this section, a recurrence relation, suitable for explicitly calculating the distribution of losses from equation (68), is presented. The relation is a form of the recurrence relation in Section A4, derived for a wider class of distributions.

A10.1 General Recurrence Relation

Suppose, in general, a power series expansion

$$G(z) = \sum_{n=0}^{\infty} A_n z^n \qquad (69)$$

defines a function *G(z)* which satisfies the differential equation

$$\frac{d}{dz}(\log G(z)) = \frac{1}{G(z)}\frac{dG(z)}{dz} = \frac{A(z)}{B(z)} \qquad (70)$$

where A and B are polynomials given respectively by

$$\begin{aligned} A(z) &= a_0 + \ldots + a_r z^r \\ B(z) &= b_0 + \ldots + b_s z^s \end{aligned} \qquad (71)$$

In other words, we require that the logarithmic derivative of G(z) be a rational function. Then, the terms of the power series expansion in equation (69) satisfy the following recurrence relation

$$A_{n+1} = \frac{1}{b_0(n+1)}\left(\sum_{i=0}^{\min(r,n)} a_i A_{n-i} - \sum_{j=0}^{\min(s-1,n-1)} b_{j+1}(n-j)A_{n-j} \right) \qquad (72)$$

To see this, rearrange the differential equation (70) as follows

$$B(z)\frac{dG}{dz} = A(z)G \qquad (73)$$

By differentiating *G* term by term, this leads to

$$\left(\sum_{j=0}^{s} b_j z^j\right)\left(\sum_{n=0}^{\infty}(n+1)A_{n+1}z^n\right) = \left(\sum_{i=0}^{r} a_i z^i\right)\left(\sum_{n=0}^{\infty} A_n z^n\right) \qquad (74)$$

For $n \geq 0$ the terms in z^n on the left hand and right hand side respectively are

$$\sum_{j=0}^{\min(s,n)} b_j (n+1-j) A_{n+1-j} \quad \text{and} \quad \sum_{i=0}^{\min(r,n)} a_i A_{n-i} \qquad (75)$$

Equating these expressions and rearranging we obtain

$$b_0(n+1)A_{n+1} = \sum_{i=0}^{\min(r,n)} a_i A_{n-i} - \sum_{j=1}^{\min(s,n)} b_j (n+1-j) A_{n+1-j} \qquad (76)$$

or equivalently

$$b_0(n+1)A_{n+1} = \sum_{i=0}^{\min(r,n)} a_i A_{n-i} - \sum_{j=0}^{\min(s-1,n-1)} b_{j+1}(n-j)A_{n-j} \qquad (77)$$

as required.

A10.2 Application

In equation (68), the probability generating function of losses was derived in the form

$$G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n}\left(\frac{1-p_k}{1-\dfrac{p_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}} \right)^{\alpha_k} \qquad (78)$$

Taking logarithmic derivatives with respect to z, it follows that

$$\frac{G'(z)}{G(z)}=\sum_{k=1}^{n}\frac{G_k'(z)}{G_k(z)}=\sum_{k=1}^{n}\frac{\dfrac{p_k\alpha_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)}\varepsilon_j^{(k)}z^{\nu_j^{(k)}-1}}{1-\dfrac{p_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}} \tag{79}$$

This expresses G'(z)/G(z) as a rational function. Accordingly, after calculation of polynomials A(z) and B(z) such that

$$\frac{A(z)}{B(z)}=\sum_{k=1}^{n}\frac{\dfrac{p_k\alpha_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)}\varepsilon_j^{(k)}z^{\nu_j^{(k)}-1}}{1-\dfrac{p_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}} \tag{80}$$

the calculation in Section A10.1 is applicable and leads to a recurrence relation for the loss amount distribution. Note that the summation described in equation (80) must be performed directly by adding the rational summands. Provided the unit size is chosen so that the exposures ν_j and therefore the degrees of numerators and denominators of the rational summands are not too large, this computation can be performed quickly.

A11 Convergence of Variable Default Rate Case to Fixed Default Rate Case

Although the CreditRisk+ Model is designed to incorporate the effects of variability in the average rates of default, there are two circumstances in which the CreditRisk+ Model behaves as if default rates were fixed. These are where:

- The standard deviation of the mean default rate for each sector tends to zero.
- The number of sectors tends to infinity.

In particular, the effect of either a large number of sectors or a low standard deviation of default rates on the portfolio is the same; the behaviour in either case is as if default rates were fixed. In the section on generalised sector analysis, this fact will be used to facilitate the analysis of specific risk within a portfolio. In this section, a proof is given of the first convergence fact. The proof of the second convergence fact is similar.

The proof proceeds by showing that the probability generating function for the CreditRisk+ Model converges to the form

$$G(z)=e^{\sum_{j=1}^{m}\left(\frac{\varepsilon_j}{\nu_j}\right)\left(z^{\nu_j}-1\right)} \tag{81}$$

which is the probability generating function in equations (17) for losses conditional on a fixed mean default rate. The CreditRisk+ Model has the following probability generating function for default losses, given at equation (68)

$$G(z)=\prod_{k=1}^{n}G_k(z)=\prod_{k=1}^{n}\left(\frac{1-p_k}{1-\dfrac{p_k}{\mu_k}\displaystyle\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}}\right)^{\alpha_k} \tag{82}$$

where $\alpha_k=\mu_k^2/\sigma_k^2$, $\beta_k=\sigma_k^2/\mu_k$, $p_k=\beta_k/(1+\beta_k)$ and $\mu_k=\displaystyle\sum_{k=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}$

We consider the limit where σ_k tends to zero. Then

$$\beta_k \to 0;\ p_k = \beta_k/(1+\beta_k) \to 0 \text{ and } \alpha_k = \mu_k/\beta_k \to \mu_k/p_k$$

Therefore

$$G(z)=\prod_{k=1}^{n}G_k(z)=\prod_{k=1}^{n}\left(\frac{1-p_k}{1-\frac{p_k}{\mu_k}\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}}\right)^{\alpha_k} \to \prod_{k=1}^{n}\left(\frac{1-p_k}{1-\frac{p_k}{\mu_k}\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}}\right)^{\frac{\mu_k}{p_k}} \tag{83}$$

In the limit

$$G(z)\to\prod_{k=1}^{n}e^{-\mu_k}e^{\sum_{j=1}^{m(k)}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}}=e^{-\sum_{j,k}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}+\sum_{j,k}\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}z^{\nu_j^{(k)}}} \tag{84}$$

On collecting terms in the exponent having common values n across different values of k, the summation over k is eliminated

$$G(z)=e^{\sum_j\frac{\varepsilon_j}{\nu_j}(z^{\nu_j}-1)} \tag{85}$$

as required.

A12 General Sector Analysis

A12.1 Introduction

In the derivation of the CreditRisk+ Model probability generating function for the distribution of losses in Section A9, it was assumed throughout that the portfolio is divided into sectors, each of which is a subset of the set of obligors. This corresponds to a situation in which obligors fall into classes, each of which is driven by one factor but all of which are mutually independent.

We now consider a more generalised situation in which, as before, a relatively small number of factors explain the systematic volatility of default rates in the portfolio, but it is not necessarily the case that the default rate of an individual obligor depends on only one of the factors. In these more general circumstances, it is not possible to describe the portfolio with sectors consisting of groupings of the obligors, but the CreditRisk+ Model incorporates this situation in the same way as before, replacing the concept of a sector with that of a systematic factor.

To understand how to generalise the sector analysis already presented, we re-examine the derivation of the probability generating function for the CreditRisk+ Model. In equation (68), the probability generating function was derived by expressing it as a product over the sectors and then integrating with respect to the distribution of default rates for each sector:

$$G(z)=\prod_{k=1}^{n}G_k(z)=\prod_{k=1}^{n}\int_{x=0}^{\infty}e^{x_k(P_k(z)-1)}f_k(x_k)dx_k \tag{86}$$

However, this expression can also be viewed as a multiple integral

$$G(z)=\int_{x_1=0}^{\infty}\ldots\int_{x_n=0}^{\infty}e^{\sum_{k=1}^{n}x_k(P_k(z)-1)}\prod_{k=1}^{n}f_k(x_k)dx_k \tag{87}$$

We regard the integrand as the probability density function of a compound Poisson distribution for any given set of values of the means x_k, $1\le k\le n$. However, we are simultaneously uncertain about all these values. Therefore, the probability density function is then integrated over the space of all possible states represented by the values of the x_k and weighted by their associated probability density functions.

Using equation (66), we can examine the exponent in the integrand in its equivalent form

$$\sum_{k=1}^{n} x_k\,(P_k(z)-1)=\sum_{k=1}^{n}\sum_{A\in k}\frac{x_k}{\mu_k}\frac{\varepsilon_A}{\nu_A}(z^{\nu_A}-1)=\sum_{A,k}\delta_{Ak}\frac{x_k}{\mu_k}\frac{\varepsilon_A}{\nu_A}(z^{\nu_A}-1) \tag{88}$$

where we have used the delta notation

$$\delta_{Ak}=\begin{cases}0 & A\notin k\\ 1 & A\in k\end{cases} \tag{89}$$

To generalise the concept of a sector in these circumstances, allowing each obligor to be influenced by more than one factor x_k, we replace the delta function with an allocation of the obligors among sectors by choosing, for each obligor A, numbers

$$\theta_{Ak}: \quad \sum_{k=1}^{n}\theta_{Ak}=1 \tag{90}$$

The allocation θ_{Ak} represents the extent to which the default probability of obligor A is affected by the factor underlying sector k. The sector analysis discussed in Section A7 corresponds to the special case

$$\theta_{Ak}=\delta_{Ak} \tag{91}$$

In the general case the expression in equation (88) is replaced by

$$\sum_{k=1}^{n} x_k\,(P_k(z)-1)=\sum_{A,k}\theta_{Ak}\frac{x_k}{\mu_k}\frac{\varepsilon_A}{\nu_A}(z^{\nu_A}-1) \tag{92}$$

where each obligor contributes a term

$$x_A(z^{\nu_A}-1) \quad \text{where} \quad x_A=\frac{\varepsilon_A}{\nu_A}\sum_{k=1}^{n}\theta_{Ak}\frac{x_k}{\mu_k} \tag{93}$$

Equation (65) is replaced by

$$P_k(z)=\frac{1}{\mu_k}\sum_{A}\theta_{Ak}\frac{\varepsilon_A}{\nu_A}z^{\nu_A} \quad \text{where} \quad \mu_k=\sum_{A}\theta_{Ak}\frac{\varepsilon_A}{\nu_A} \tag{94}$$

A12.2 Performing the Sector Decomposition

In this section, we show how to assimilate the data for the CreditRisk+ Model for generalised sector analysis. We assume that for each obligor in the portfolio an estimate has been made of the extent to which the volatility of the obligor's default rate is explained by the factor *k*. As explained in Section A12.1, this is expressed by a choice of number

$$\theta_{Ak}: \quad \sum_{k=1}^{n}\theta_{Ak}=1 \tag{95}$$

for each sector k and obligor A in the portfolio. The number θ_{Ak} represents our judgement of the extent to which the state of sector *k* influences the fortunes of obligor A.

As in the special case discussed in Section A7, we must also provide estimates of the mean and standard deviation for each sector. We indicate a method of estimating these parameters, assuming again that estimates have been obtained of both quantities for each obligor by reference to obligor credit quality.

The mean for each sector is the sum of contributions from each obligor, but now weighted by the allocations θ_{Ak}. Thus

$$\mu_k = \sum_A \theta_{Ak} \mu_A \tag{96}$$

Then, by analogy with equation (43), we express the ratio σ_k / μ_k as a weighted average of contributions from each obligor

$$\frac{\sigma_k}{\mu_k} = \frac{\sum_A \theta_{Ak} \mu_A \dfrac{\sigma_A}{\mu_A}}{\sum_A \theta_{Ak} \mu_A} \quad \text{;hence} \quad \sigma_k = \sum_A \theta_{Ak} \sigma_A \tag{97}$$

This estimates the standard deviation for each factor. The discussion in Section A7 is recaptured when $\theta_{Ak} = \delta_{Ak}$ as discussed above.

A12.3 Incorporating Specific Factors

So far we have assumed all variability in default rates in the portfolio to be systematic. Potentially, we require an additional sector to model factors specific to each obligor.

However, specific factors can be modelled without recourse to a large number of sectors. It was remarked in Section A11 that assigning a zero variance to a sector is equivalent to assuming that the sector is itself a portfolio composed of a large number of sub-sectors. Hence, for a portfolio containing a large number of obligors, only one sector is necessary in order to incorporate specific factors. Let the specific factor sector be sector 1. Then, for each obligor A, the proportion of the variance of the expected default frequency for that obligor that is explained by specific risk is θ_{A1}. Sector 1 would be assigned a total standard deviation given by equation (97). However, for the specific factor sector only, this standard deviation can be set to zero. The specific factor sector then behaves as the limit of a large number of sectors, one for each obligor in the portfolio, with independent variability of their default rate. The lost standard deviation represented by σ_1 is a measure of the benefit of the presence of specific factors in the portfolio.

A13 Risk Contributions and Pairwise Correlation

A13.1 Introduction

In this section, we derive formulae for two useful measures connected with the default loss distribution, as follows:

- Risk contributions are defined as the contributions made by each obligor to the unexpected loss of the portfolio, measured either by a chosen percentile level or the standard deviation.
- Pairwise correlations between default events give a measure of the extent to which concentration risk is present in the portfolio.

A13.2 Risk Contributions

In this section, we derive a formula for the contribution of an individual obligor to the standard deviation of the loss distribution in the CreditRisk+ Model.

For a portfolio of obligors A having exposure E_A, the risk contribution for obligor A can be defined as the marginal effect of the presence of E_A on the standard deviation of the distribution of credit losses. Alternatively, the risk contribution can be defined as the marginal effect of the presence of E_A on some other measure of portfolio aggregate risk, such as a given loss percentile.

In the first case, an analytic formula for the risk contribution is possible. The risk contribution can be written

$$RC_A = E_A \frac{\partial \sigma}{\partial E_A} \text{ , or equivalently } RC_A = \frac{E_A}{2\sigma} \frac{\partial \sigma^2}{\partial E_A} \tag{98}$$

Moreover, for most models including the CreditRisk+ Model, the risk contributions defined by equation (98) add up to the standard deviation. This is because of the variance-covariance formula

$$\sigma^2 = \sum_{A,B} \rho_{AB} E_A E_B \sigma_A \sigma_B \tag{99}$$

where σ_A and σ_B are the standard deviations of the default event indicator for each obligor. Provided the model is such that the correlation coefficients are independent of the exposures, equation (99) expresses the variance as a homogeneous quadratic polynomial in the exposures. Hence, by a general property of homogeneous polynomials we have

$$\sum_A RC_A = \frac{1}{2\sigma} \sum_A E_A \frac{\partial \sigma^2}{\partial E_A} = \frac{2\sigma^2}{2\sigma} = \sigma \tag{100}$$

If the marginal effect on a given percentile is chosen as the definition of risk contributions, then an analytic formula will not be possible. Instead, one can use the approximation described next.

Let ε, σ and X be the expected loss, the standard deviation of losses and the loss at a given percentile level from the distribution. Define the multiplier to the given percentile as ξ where

$$\varepsilon + \xi\sigma = X \tag{101}$$

Then, we can define risk contributions to the percentile in terms of the contributions to the standard deviation by writing

$$\hat{RC}_A = \varepsilon_A + \xi RC_A \tag{102}$$

Then, in view of equations (100) and (101), we have

$$\sum_A \hat{RC}_A = \sum_A (\varepsilon_A + \xi RC_A) = \varepsilon + \xi\sigma = X \tag{103}$$

In the analysis below, we will concentrate on the determination of the contributions to the standard deviation. In order to evaluate the right hand side of equation (98), we derive analytic formulae for mean and variance of the distributions of default events and default losses in the CreditRisk+ Model. We use the following definitions, referring to a sector *k*, which are consistent with the notation used previously. Since the mean and variance of the distribution of losses in the CreditRisk+ Model are both additive across sectors, we can work with one sector for most of the analysis. For ease of notation, we have therefore suppressed the reference to sector *k* where it is not necessary.

Reference	Symbol	Mean	Variance
Loss severity polynomial (equation 94)	$P(z)$		
Default event probability generating function conditional on mean x	$E(z,x)$	μ_E	σ_E^2
Probability density function for mean x	$f(x)$	μ_f	σ_f^2
Default event probability generating function	$F(z)$	$\mu_F = \mu_k$	σ_F^2
CreditRisk+ Model probability generating function	$G(z)$	$\mu_G = \varepsilon_k$	σ_G^2

Here μ_k and ε_k are the mean number of default events in sector *k* and the expected loss from sector *k* respectively. We have

$$G(z) = F(P(z)) \tag{104}$$

This is merely a restatement of equation (64). Also, by equation (53)

$$F(z) = \int_x E(z,x) f(x) dx \tag{105}$$

For the probability generating functions E, F and G, we have, by general properties of probability generating functions

$$\mu_E = \frac{dE}{dz}(1) \ , \quad \mu_F = \frac{dF}{dz}(1) \quad \text{and} \quad \mu_G = \frac{dG}{dz}(1) \tag{106}$$

$$\sigma_E^2 + \mu_E^2 = \frac{d^2E}{dz^2}(1) + \frac{dE}{dz}(1) \ , \quad \sigma_F^2 + \mu_F^2 = \frac{d^2F}{dz^2}(1) + \frac{dF}{dz}(1) \quad \text{and}$$
$$\sigma_G^2 + \mu_G^2 = \frac{d^2G}{dz^2}(1) + \frac{dG}{dz}(1) \tag{107}$$

By definition of x, we have

$$\mu_E(x) = x \tag{108}$$

Because E(z, x) is the probability generating function of a Poisson distribution, we also have

$$\sigma_E^2 = \mu_E \tag{109}$$

By equations (105) and (106), bringing the differentiation by the auxiliary variable z under the integration sign, we obtain

$$\mu_F = \int_x \mu_E(x) f(x) dx = \int_x x f(x) dx = \mu_f \tag{110}$$

Similarly, using equations (105) and (107)

$$\sigma_F^2 + \mu_F^2 = \int_x \left(\sigma_E^2 + \mu_E^2\right) f(x) dx = \int_x \left(\mu_E + \mu_E^2\right) f(x) dx = \mu_f + \sigma_f^2 + \mu_f^2 \tag{111}$$

Hence

$$\sigma_F^2 = \mu_f + \sigma_f^2 \tag{112}$$

Equations (110) and (112) are the mean and variance of the distribution of default events. To provide the link to the moments of the loss distribution, we use equation (104), which yields, by the chain rule

$$\frac{dG}{dz}(z) = \frac{dF}{dz}(P(z))\frac{dP}{dz} \ ; \quad \frac{d^2G}{dz^2}(z) = \frac{d^2F}{dz^2}(P(z))\left(\frac{dP}{dz}\right)^2 + \frac{dF}{dz}(P(z))\frac{d^2P(z)}{dz^2} \tag{113}$$

Hence

$$\sigma_G^2 = \frac{d^2F}{dz^2}(P(1))\frac{dP}{dz}(1)^2 + \frac{dF}{dz}(P(1))\frac{d^2P}{dz^2}(1) + \frac{dF}{dz}(P(1)).\frac{dP}{dz}(1) - \left(\frac{dF}{dz}(P(1)).\frac{dP}{dz}(1)\right)^2 \tag{114}$$

Successive differentiation of equation (94) yields

$$P(1)=1\ ;\ \frac{dP}{dz}(1)=\frac{1}{\mu_k}\sum_A \theta_{Ak}\varepsilon_A=\frac{\varepsilon_k}{\mu_k}\quad\text{and}\quad\frac{d^2P}{dz^2}(1)=\frac{1}{\mu_k}\sum_A \theta_{Ak}\varepsilon_A(\nu_A-1) \tag{115}$$

On substituting equations (112) and (115) into equation (114), we obtain

$$\sigma_G^2=\left(\sigma_F^2+\mu_k^2-\mu_k\right)\left(\frac{1}{\mu_k}\sum_A \theta_{Ak}\varepsilon_A\right)^2+\sum_A \theta_{Ak}\varepsilon_A\nu_A-\left(\sum_A \theta_{Ak}\varepsilon_A\right)^2 \tag{116}$$

Substituting for ε_k, we obtain

$$\sigma_G^2=\left(\sigma_k^2+\mu_k^2\right)\left(\frac{\varepsilon_k}{\mu_k}\right)^2+\sum_A \theta_{Ak}\varepsilon_A\nu_A-\varepsilon_k^2=\sigma_k^2\left(\frac{\varepsilon_k}{\mu_k}\right)^2+\sum_A \theta_{Ak}\varepsilon_A\nu_A \tag{117}$$

Finally, summing over sectors gives the standard deviation of the CreditRisk+ Model Loss Distribution for the whole portfolio

$$\sigma^2=\sum_{k=1}^{n}\varepsilon_k^2\left(\frac{\sigma_k}{\mu_k}\right)^2+\sum_A \varepsilon_A\nu_A \tag{118}$$

Note that this is the standard deviation of the actual distribution of losses. As in the earlier sections, the σ_k denote the standard deviations of the factors driving the default rates in each sector.

Risk contributions can now be derived directly by differentiating equation (118). Thus, by equation (98)

$$RC_A=E_A\frac{\partial\sigma}{\partial E_A}=\frac{E_A}{2\sigma}\frac{\partial\sigma^2}{\partial E_A} \tag{119}$$

Hence

$$RC_A=\frac{E_A}{2\sigma}\frac{\partial}{\partial E_A}\left(\sum_B \varepsilon_B\nu_B+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\varepsilon_k^2\right)=\frac{E_A}{2\sigma}\left(2E_A\mu_A+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2 2\varepsilon_k\theta_{Ak}\mu_A\right) \tag{120}$$

where we have interchanged E_A and ν_A for notational convenience. Hence

$$RC_A=\frac{E_A\mu_A}{\sigma}\left(E_A+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\varepsilon_k\theta_{Ak}\right) \tag{121}$$

This is the required formula for risk contributions to the standard deviation. As remarked above, it can be shown explicitly that the risk contributions add up to the standard deviation of the portfolio loss distribution. Thus, from equation (121),

$$\sum_A RC_A=\sum_A\frac{E_A\mu_A}{\sigma}\left(E_A+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\theta_{Ak}\varepsilon_k\right)=\sum_A\frac{E_A^2\mu_A}{\sigma}+\sum_A\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\frac{E_A\mu_A}{\sigma}\theta_{Ak}\varepsilon_k \tag{122}$$

Hence, using equation (118)

$$\sum_A RC_A=\sum_A\frac{\varepsilon_A\nu_A}{\sigma}+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\varepsilon_k\sum_A\theta_{Ak}\frac{E_A\mu_A}{\sigma}=\frac{1}{\sigma}\left(\sum_A\varepsilon_A\nu_A+\sum_k\left(\frac{\sigma_k}{\mu_k}\right)^2\varepsilon_k^2\right)=\frac{\sigma^2}{\sigma}=\sigma \tag{123}$$

as required.

A13.3 Pairwise Correlation

In this section, we derive a formula for the pairwise correlation between default events in the CreditRisk+ Model. To define carefully the pairwise correlation over a time period Δt, we associate to each obligor its indicator function I_A, which is the random variable having the values

$$I_A = \begin{cases} 1 \text{ if obligor A defaults in the time period} \\ 0 \text{ otherwise} \end{cases} \tag{124}$$

Then, the correlation ρ between default of two obligors A and B in the time period Δt is defined by

$$\rho_{AB} = \rho(I_A, I_B) \tag{125}$$

That is, the statistical correlation between the indicator functions of A and B in the time period. If the expected values of I_A, I_B and the product I_{AB} are μ_A, μ_B and μ_{AB}, respectively, then μ_A, μ_B and μ_{AB} are, respectively, the expected number of defaults of A, B and of both obligors in the time period. Then, because the indicator functions can only take on the values 0 or 1, the standard expression for correlation reduces to the following form:

$$\rho_{AB} = \frac{\mu_{AB} - \mu_A \mu_B}{(\mu_A - \mu_A^2)^{1/2} (\mu_B - \mu_B^2)^{1/2}} \tag{126}$$

We seek an expression for the right hand side of equation (126) in the context of the CreditRisk+ Model. We take two distinct obligors A and B and make the following definitions, where the general sector decomposition is used with n sectors.

Reference	Obligor A	Obligor B
Time period	Δt	
Instantaneous default probability	p_A	p_B
Expected number of defaults	$\mu_A = 1 - e^{-p_A \Delta t} \approx p_A \Delta t$	$\mu_B = 1 - e^{-p_B \Delta t} \approx p_B \Delta t$
Sector decomposition	$\theta_{AK}; 1 \le k \le n$	$\theta_{BK}; 1 \le k \le n$

The unknown term in equation (126) is the expected joint default expectation μ_{AB} . Since A and B are distinct, for any realised values of the sector means x_k, $1 \le k \le n$ the events of default are independent, we have, writing x_A and x_B as in equation (93)

$$\mu_{AB} = \int_{x_1} \dots \int_{x_n} x_A x_B \prod_{k=1}^{n} f_k(x_k) dx_k \tag{127}$$

where, as shown in the table, we have approximated the integrand, ignoring higher powers of the default expectations and using the following approximation[6]

$$(1 - e^{-x_A \Delta t})(1 - e^{-x_B \Delta t}) \approx x_A x_B \tag{128}$$

In view of the sector decomposition, we have, by equation (93)

$$x_A = \sum_{k=1}^{n} \frac{x_k}{\mu_k} \theta_{Ak} \mu_A \quad \text{and} \quad x_B = \sum_{k=1}^{n} \frac{x_k}{\mu_k} \theta_{Bk} \mu_B \tag{129}$$

For convenience, define coefficients $\omega_{kk'}$ by writing

$$\omega_{kk'} = \frac{\theta_{Ak} \theta_{Bk'}}{\mu_k \mu_{k'}} \mu_A \mu_B \tag{130}$$

[6] The integration in (127) can be performed without using the approximation (128) to give an exact value for the correlation. It is interesting to note that, while the exact integration, which has a similar form to equation (54), depends on knowledge of the probability generating function *f*, only the mean and standard deviation of *f* are required to estimate the approximate correlation given by equation (138). In this sense the choice of gamma distribution for the variability of the mean default rate is irrelevant to correlation considerations.

Then

$$\mu_{AB} = \int_{x_1} \cdots \int_{x_n} \sum_{k,k'} \omega_{kk'} x_k x_{k'} \prod_{k=1}^{n} f_k(x_k)\,dx_k \tag{131}$$

and we deduce that

$$\begin{aligned}\mu_{AB} = &\sum_{k \neq k'} \omega_{kk'} \iint_{x_k, x_{k'}} x_k x_{k'} f_k(x_k) f_k(x_{k'})\,dx_k\,dx_{k'} \int_{x_j; j \neq k,k'} \prod_{j=1; j \neq k,k'}^{n} f_j(x_j)\,dx_j \\ &+ \\ &\sum_{k=1}^{n} \omega_{kk} \int_{x_k} x_k^2 f_k(x_k)\,dx_k \int_{x_j; j \neq k} \prod_{j=1; j \neq k}^{n} f_j(x_j)\,dx_j\end{aligned} \tag{132}$$

Hence

$$\mu_{AB} = \sum_{k,k'=1; k \neq k'}^{n} \omega_{kk'} \mu_k \mu_{k'} + \sum_{k=1}^{n} \omega_{kk} \left(\mu_k^2 + \sigma_k^2\right) \tag{133}$$

or

$$\mu_{AB} = \sum_{k,k'=1; k \neq k'}^{n} \omega_{kk'} \mu_k \mu_{k'} + \sum_{k=1}^{n} \omega_{kk} \left(\mu_k^2 + \sigma_k^2\right) \tag{134}$$

However

$$\sum_{k,k'=1}^{n} \omega_{kk'} \mu_k \mu_{k'} = \left(\sum_{k=1}^{n} \frac{\theta_{Ak}\mu_A}{\mu_k} \mu_k\right)\left(\sum_{k=1}^{n} \frac{\theta_{Bk}\mu_B}{\mu_k} \mu_k\right) = \mu_A \mu_B \tag{135}$$

Thus

$$\mu_{AB} = \mu_A \mu_B + \sum_{k=1}^{n} \omega_{kk} \sigma_k^2 \tag{136}$$

Substituting for ω_{kk}, we obtain

$$\rho_{AB} = \frac{\mu_{AB} - \mu_A \mu_B}{(\mu_A - \mu_A^2)^{1/2} (\mu_B - \mu_B^2)^{1/2}} = (\mu_A \mu_B)^{-1/2} \sum_{k=1}^{n} \theta_{Ak} \theta_{Bk} \mu_A \mu_B \left(\frac{\sigma_k}{\mu_k}\right)^2 \tag{137}$$

This simplifies to

$$\rho_{AB} = (\mu_A \mu_B)^{1/2} \sum_{k=1}^{n} \theta_{Ak} \theta_{Bk} \left(\frac{\sigma_k}{\mu_k}\right)^2 \tag{138}$$

Equation (138) is the formula for default event correlation between distinct obligors A and B in the CreditRisk+ Model. Equation (138) is valid wherever the likely probabilities of default over the time period in question are small, taking into account their standard deviation. It is not universally valid. Note, in particular, that the value of ρ_{AB} given by the formula can be more than one if too large values of the means and standard deviations are chosen. This corresponds to the approximation used at equation (128) - the left-hand side is clearly always less than unity while the approximating function is unbounded.

We note two salient features of equation (138):

- If the obligors A and B have no sector in common then the correlation between them will be zero. This is because no systematic factor affects them both.
- If it is accepted that, as suggested by historical data, the ratios σ_k/μ_k are of the order of unity, then depending on the sector decomposition the correlation has the same order as the term $\sqrt{(\mu_A \mu_B)}$ in the equation. This is the geometric mean of the two default probabilities. Therefore, in general one would expect default correlations to typically be of the same order of magnitude as default probabilities themselves.

Appendix B - Illustrative Example

B

B1 Example Spreadsheet-Based Implementation

The purpose of this appendix is to illustrate the application of the CreditRisk+ Model to an example portfolio of exposures with the use of a spreadsheet-based implementation of the model.

The implementation, consisting of a single spreadsheet together with an addin, can be downloaded from the Internet (http://www.csfb.com) to reproduce the results shown in this appendix. The spreadsheet contains three examples of the use of the CreditRisk+ Model. In addition, the spreadsheet can be used on a user-defined portfolio.

For illustrative purposes, we have limited the example portfolio size to only 25 obligors. However, the spreadsheet implementation has been designed to allow analysis of portfolios of realistic size. Up to 4,000 individual obligors and up to 8 sectors can be handled by the spreadsheet implementation. However, there is no limit, in principle, to the number of obligors that can be handled by the CreditRisk+ Model. Increasing the number of obligors has only a limited impact on the processing time.

B2 Example Portfolio and Static Data

The three examples are based on a portfolio consisting of 25 obligors of varying credit quality and size of exposure. The exposure amounts are net of recovery. Details of this portfolio are given in Table 8 opposite.

◀ Table 8:
Example portfolio

Name	Exposure	Credit Rating
1	358,475	H
2	1,089,819	H
3	1,799,710	F
4	1,933,116	G
5	2,317,327	G
6	2,410,929	G
7	2,652,184	H
8	2,957,685	G
9	3,137,989	D
10	3,204,044	D
11	4,727,724	A
12	4,830,517	D
13	4,912,097	D
14	4,928,989	H
15	5,042,312	F
16	5,320,364	E
17	5,435,457	D
18	5,517,586	C
19	5,764,596	E
20	5,847,845	C
21	6,466,533	H
22	6,480,322	H
23	7,727,651	B
24	15,410,906	F
25	20,238,895	E

The example uses a credit rating scale to assign default rates and default rate volatilities to each obligor. A table giving an example mapping from credit ratings to a set of default rates and default rate volatilities is given. The table is shown as Table 9 below. The credit rating scale and other data in the table are designed for the purposes of the example only.

◀ Table 9:
Example mapping table of default rate information

Credit Rating	Mean Default rate	Standard Deviation
A	1.50%	0.75%
B	1.60%	0.80%
C	3.00%	1.50%
D	5.00%	2.50%
E	7.50%	3.75%
F	10.00%	5.00%
G	15.00%	7.50%
H	30.00%	15.00%

B3 Example use of the Spreadsheet Implementation

Three examples are given, each based on the same portfolio, as follows:

- All obligors are allocated to a single sector.
- Each obligor is allocated to only one sector. In this example, countries are the sectors. This assumes that each obligor is subject to only one systematic factor, which is responsible for all of the uncertainty of the obligor's default rate.
- Each obligor is apportioned to a number of sectors. Again, countries are the sectors. This reflects the situation in which the fortunes of an obligor are affected by a number of systematic factors.

The examples are installed on the spreadsheet implementation, together with the results generated by the model. For each example, the inputs to the model have been set to generate the following:

- Percentiles of loss.
- Full loss distribution.
- Credit risk provision.
- Risk contributions.

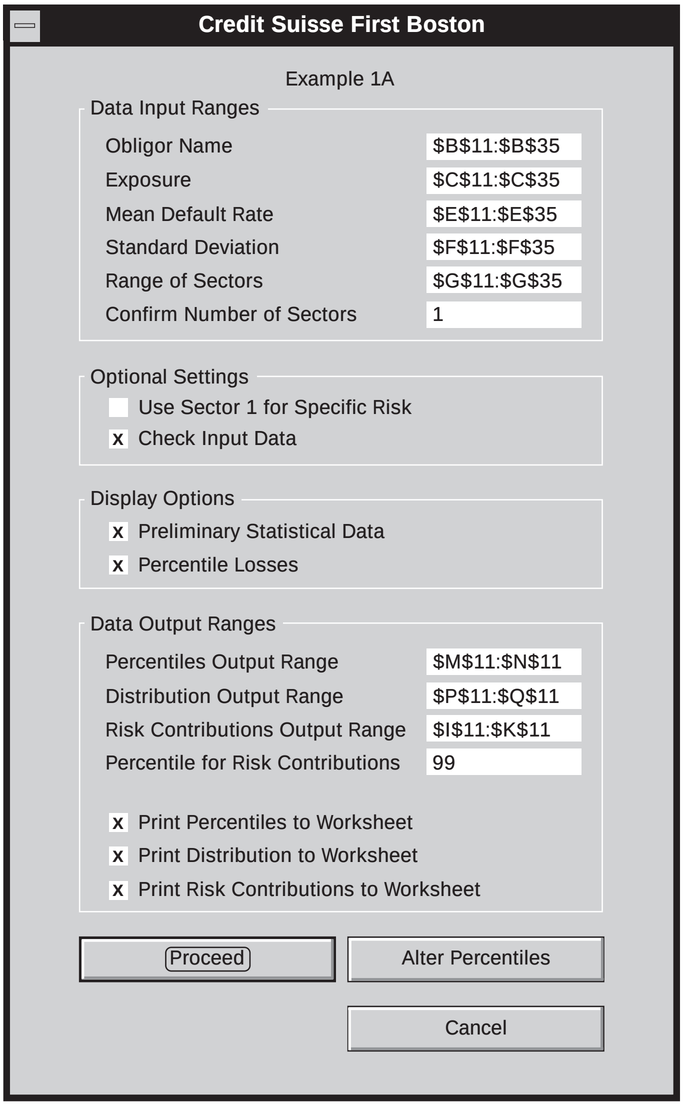
In this section, the steps to reproducing these results using the model implementation are described.

B3.1 Activating the CreditRisk$^+$ Model

Choose one of the three example worksheets to reproduce the results. Each worksheet is equipped with a macro button. Press the button to activate the model implementation.

B3.2 Data Input Screen

On activation, the model will show the Data Input Screen. This screen is used to set the worksheet ranges of data to be read in to the model and to specify the form of output data required. The Data Input Screen has been preset to the correct ranges corresponding to the layout of each example worksheet.



◀ Data Entry Screen

Press the Proceed button on the Data Entry Screen to proceed to the next step.

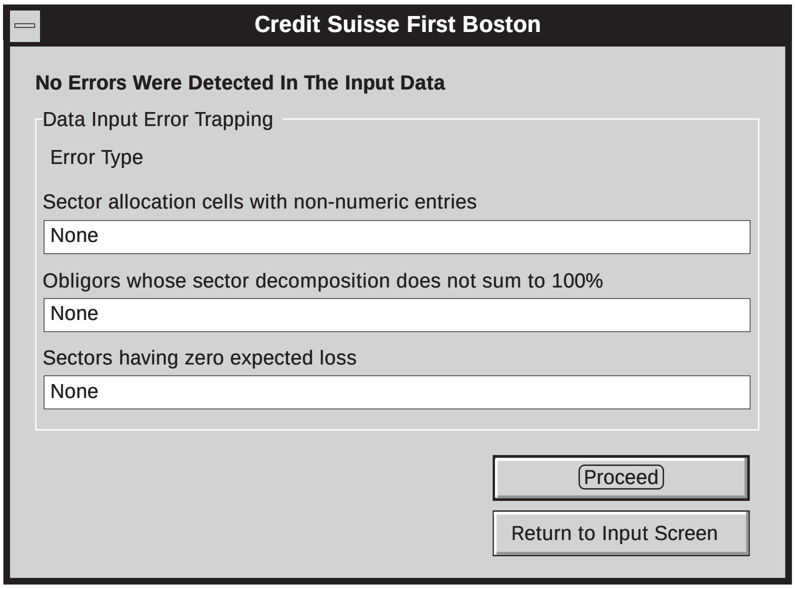
Press the Proceed button on the Data Entry Screen to proceed to the next step.

B3.3 Input Data Check

The model implementation has been preset to identify errors in the data read in before the calculation commences. The user is given the option of switching off this facility via the Data Input Screen. The model implementation ensures that the data satisfies the following three criteria:

- The sector allocation table contains only numeric data.
- The decomposition of each obligor to the various sectors adds up to 100%.
- A sector must contain at least one allocation entry.

The Input Data Check screen indicates the location of an error in the sector allocation table.

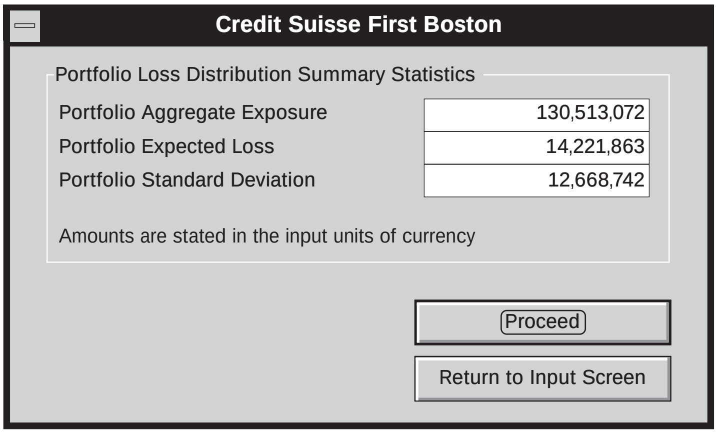


▶ Input Data Check Screen
Press the Proceed button on the Input Data Check Screen to proceed with the calculation.

Press the Proceed button on the Input Data Check Screen to proceed with the calculation.

B3.4 Portfolio Loss Distribution Summary Statistics

The model implementation has been preset to display summary statistics of the portfolio loss distribution. The user is given the option to switch off this facility via the Data Input Screen.

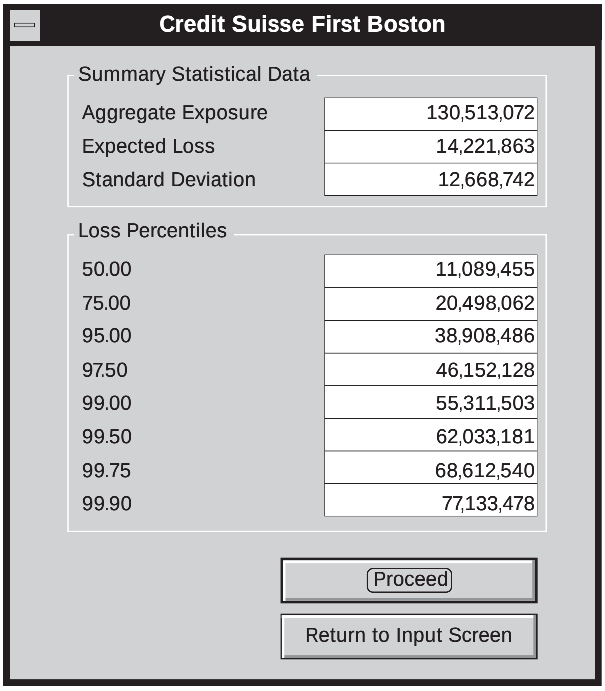


▶ Input Data Check Screen
Press the Proceed button to proceed to the next step. The model implementation now calculates the full distribution of losses.

Press the Proceed button to proceed to the next step. The model implementation now calculates the full distribution of losses.

B3.5 Percentile Losses, Loss Distribution and Credit Risk Provision

The model implementation has been preset to display summary statistics of the portfolio loss distribution. The user can switch off this facility via the Data Input Screen.



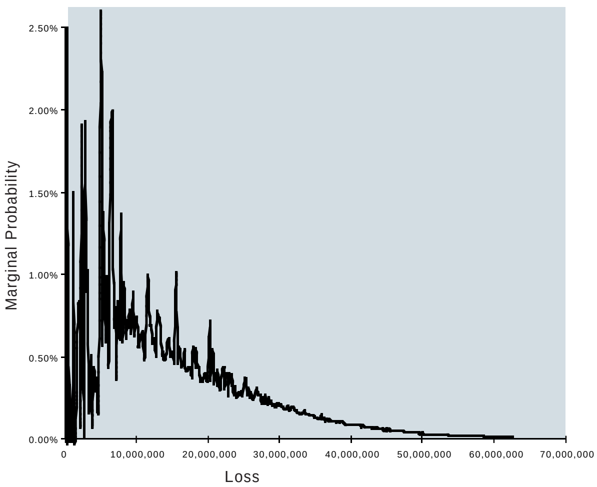
◀ Input Data Check Screen

Press the Proceed button to output the loss percentiles, the loss distribution, and the risk contributions to the worksheet.

Press the Proceed button to output the loss percentiles, the loss distribution, and the risk contributions to the worksheet.

Loss Distribution

A graph of the loss distribution has been set up on each worksheet using the results generated from the step above.



◀ Credit Loss Distribution

Credit Risk Provision

From the summary statistic data above, the Annual Credit Provision (ACP) is given by the Expected Loss, i.e. 14,221,863. If the 99th percentile level is chosen as the determining level for the Incremental Credit Reserve Cap (ICR Cap), then the ICR Cap is 55,311,503.

B3.6 Risk Contributions

In each example, the model implementation has been preset to output risk contributions for each obligor in the example portfolio. The risk contribution calculated by the model is defined as the marginal impact of the obligor on a chosen percentile of the loss distribution. The model implementation has been preset to calculate risk contributions by reference to the 99th percentile loss. This setting can be altered to a different percentile via the Data Entry Screen.

B3.7 Using Risk Contributions For Portfolio Management

Example 1 has been split into two examples, 1A and 1B, to illustrate the use of risk contributions in portfolio management as follows:

- In example 1A, all 25 obligors are included in the portfolio. Table 10 shows that obligors 24 and 25 have the largest risk contributions.
- In example 1B, obligors 24 and 25 have been removed from the portfolio. The other portfolio data is unchanged.

The risk contribution output from example 1A is repeated in the table below.

Table 10:
Example 1A Risk Contributions

Name	Expected Loss	Risk Contribution
1	107,543	228,711
2	326,946	764,758
3	179,971	426,743
4	289,967	716,735
5	347,599	896,874
6	361,639	910,914
7	795,655	2,163,988
8	443,653	1,199,910
9	156,899	434,047
10	160,202	437,350
11	70,916	225,356
12	241,526	756,325
13	245,605	794,754
14	1,478,697	4,773,594
15	504,231	1,602,530
16	399,027	1,330,448
17	271,773	892,720
18	165,528	560,564
19	432,345	1,477,654
20	175,435	593,559
21	1,939,960	6,850,969
22	1,944,097	7,110,748
23	123,642	487,938
24	1,541,091	9,056,197
25	1,517,917	10,618,120

The effect on the test portfolio of removing obligors 24 and 25 is shown in table 12 below. Removing these obligors in example 1B has two effects on the portfolio:

- The expected loss of the portfolio has been reduced by 3,059,008 from 14,221,863 to 11,162,856. The amount of expected loss removed is exactly equal to the expected losses from the two removed obligors because expected loss is additive across the portfolio. Thus the ACP provision in respect of the portfolio can be reduced by 3,059,008.
- The 99th percentile loss from the portfolio has declined by 15,364,646 from 55,311,503 to 39,946,857. This is approximately predicted by the total risk contributions of 19,674,317 from the two obligors removed. The risk contributions give an estimate of the effect of removing the obligors. Thus, if the 99th percentile loss is used as the ICR Cap for the portfolio, then the ICR Cap can be reduced by 15,364,646. Furthermore, if the same percentile is used as the benchmark confidence level for determining economic capital, then the amount of economic capital required to support the portfolio is reduced by the same amount.

The tables below summarise the portfolio movement and the risk details of the removed obligors.

Table 11: Example 1B - Risk analysis of removed obligors

Name	Exposure	Expected Loss	Risk Contribution
24	15,410,906	1,541,091	9,056,197
25	20,238,895	1,517,917	10,618,120
Total	35,649,801	3,059,008	19,674,317

Table 12: Example 1B - Portfolio movement analysis

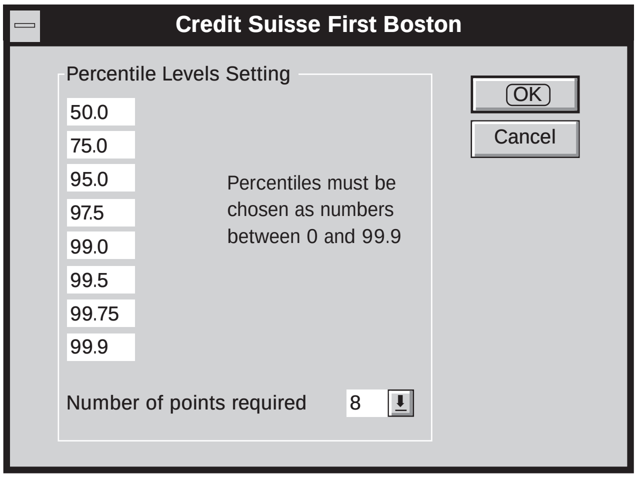
	Example 1A	Example 1B	Absolute Movement	% Movement
Exposure	130,513,072	94,863,271	(35,649,801)	27.3%
Mean	14,221,863	11,162,856	(3,059,007)	21.5%
99th Percentile	55,311,503	39,946,857	(15,364,646)	27.8%

Although the example incorporates unrealistic levels of default rates, the percentage movements in Table 12 illustrate a general feature of portfolio risk management. The removal of the obligors with the largest risk contributions from a portfolio has a greater impact on the portfolio risk, as measured by the 99th or percentile loss, than on the expected loss of the portfolio. Therefore, a significant reduction in the economic capital required to support a portfolio of credit exposures can be achieved by focusing on the management of a small number of obligors with large risk contributions.

Thus in this case, removal of two obligors, representing 21.5% of the expected loss of the portfolio, has eliminated 27.8% of the total risk as measured by the 99th percentile loss.

B3.8 Setting the Percentile Levels

The model implementation provides a facility to change the percentile loss levels calculated and output by the model. This facility is accessed from the Data Entry Screen.



Data Entry Screen

Appendix C - Contacts

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Appendix D - Selected Bibliography

D

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