Lemma 1 *A semimartingale X is a special semimartingale if and only if $\sqrt{[X]} \in \mathcal{A}_{loc}$.*

Proof: Let X be a special semimartingale and let $X = X(0) + L + A$ be a decomposition of X. Obviously[1]

$$\begin{aligned}
\sqrt{[X]} &\leq \sqrt{[L]} + \sqrt{[A]} = \\
&= \sqrt{[L]} + \sqrt{\sum (\Delta A)^2} \leq \\
&\leq \sqrt{[L]} + \sum \sqrt{(\Delta A)^2} = \\
&= \sqrt{[L]} + \sum |\Delta A| \leq \\
&\leq \sqrt{[L]} + \operatorname{Var}(A).
\end{aligned}$$

As X is a special semimartingale[2] $\operatorname{Var}(A) \in \mathcal{A}_{\text{loc}}$. As L is a local martingale[3] $\sqrt{[L]} \in \mathcal{A}_{\text{loc}}$, so $\sqrt{[X]} \in \mathcal{A}_{\text{loc}}$. On the other hand assume now, that $\sqrt{[X]} \in \mathcal{A}_{\text{loc}}$. Let $X = X(0) + L + V$ be a decomposition of X, where $V \in \mathcal{V}$. To show that X is a special semimartingale one should show that $\operatorname{Var}(V) \in \mathcal{A}_{\text{loc}}$. Obviously[4]

$$\begin{aligned}
\sqrt{[V]} &= \sqrt{[X - L]} \leq \sqrt{[X]} + \sqrt{[-L]} = \\
&= \sqrt{[X]} + \sqrt{[L]}.
\end{aligned}$$

As L is a local martingale $\sqrt{[L]} \in \mathcal{A}_{\text{loc}}$ so $\sqrt{[V]} \in \mathcal{A}_{\text{loc}}$. Let V^c be the continuous and V^d be the discontinuous part of V. Obviously

$$\operatorname{Var}(V) = \operatorname{Var}\left(V^c + V^d\right) \leq \operatorname{Var}(V^c) + \operatorname{Var}\left(V^d\right).$$

$\operatorname{Var}(V^c)$ is continuous so it is locally bounded, hence $\operatorname{Var}(V^c) \in \mathcal{A}_{\text{loc}}$. Therefore it is sufficient[5] to prove the

$$\operatorname{Var}\left(V^d\right) = \sum |\Delta V| \in \mathcal{A}_{\text{loc}}.$$

From the condition $\sqrt{[V]} \in \mathcal{A}_{\text{loc}}$ obviously

$$\sqrt{\sum \left(\Delta V^d\right)^2} = \sqrt{[V]} \in \mathcal{A}_{\text{loc}}.$$

One should prove that the condition $\sqrt{\sum \left(\Delta V^d\right)^2} \in \mathcal{A}_{\text{loc}}$ implies that $\sum \left|\Delta V^d\right| \in \mathcal{A}_{\text{loc}}$. As $(\mathcal{A}_{\text{loc}})_{\text{loc}} = \mathcal{A}_{\text{loc}}$ one can assume, that $\sqrt{[V]} \in \mathcal{A}$. If

$$\tau_n \mathrel{\hat{=}} \inf\{t \mid \operatorname{Var}(V(t)) \geq n\},$$

[1] Corollary 2.36, page 137.
[2] Theorem 4.44. page 257.
[3] Theorem 3.62. page 222.
[4] Corollary 2.36. page 137.
[5] Theorem 4.44. page 257.

then as $\sqrt{[V]} \in \mathcal{A}$

$$\begin{aligned}\operatorname{Var}\left(V^{d}\right)^{\tau_{n}} &\leq n+\Delta \operatorname{Var}\left(V^{d}\left(\tau_{n}\right)\right)= \\ &= n+\left|\Delta V^{d}\left(\tau_{n}\right)\right| \leq n+\sqrt{\left(\Delta V^{d}\left(\tau_{n}\right)\right)^{2}} \leq \\ &\leq n+\sqrt{[V](\infty)} \in L^{1}(\Omega),\end{aligned}$$

so $V^{d} \in \mathcal{A}_{\text{loc}}$.

□

Now we give a simple proof of Theorem 4.49. Let X be a special semimartingale and let $X = X(0) + A + L$ be the canonical decomposition of X. Assume that $H \bullet X$ exists and it is a special semimartingale. The integrability means that for some decomposition $X = X(0) + V + M$

$$H \bullet X = H \bullet V + H \bullet M.$$

With simple calculation

$$\begin{aligned}\sqrt{H^{2} \bullet [L]} &\leq \sqrt{H^{2} \bullet [X]}+\sqrt{H^{2} \bullet [A]}= \\ &= \sqrt{H^{2} \bullet [X]}+\sqrt{\sum (H \Delta A)^{2}} \leq \\ &\leq \sqrt{H^{2} \bullet [X]}+\sum |H \Delta A| \leq \\ &\leq \sqrt{H^{2} \bullet [X]}+\operatorname{Var}(H \bullet A).\end{aligned}$$

By the assumptions of the theorem $H \bullet X$ is a special semimartingale so

$$\sqrt{[H \bullet X]}=\sqrt{H^{2} \bullet [X]} \in \mathcal{A}_{\text{loc}}^{+}.$$

As X is a special semimartingale obviously[6] $V \in \mathcal{A}_{\text{loc}}$ and as $V = A + N$, where N is a local martingale[7]

$$V^{p}=(A+N)^{p}=A^{p}+0=A.$$

Let us also observe that[8]

$$H \bullet A=H \bullet V^{p}=(H \bullet V)^{p} \in \mathcal{A}_{\text{loc}}.$$

Hence $\operatorname{Var}(H \bullet A) \in \mathcal{A}_{\text{loc}}$, so

$$\sqrt{H^{2} \bullet [L]} \in \mathcal{A}_{\text{loc}}^{+},$$

therefore $H \bullet M$ exists. As integral is obviously linear in the integrator the integral $H \bullet A$ is also well defined.

□

[6] Theorem 4.44. page 257.

[7] See: page 217, first, second and third properties of predictable compensator.

[8] See: page 217, seventh property of predictable compensator.