The Measurable Projection and Selection Theorems

The goal of these notes is to give a simple and straightforward proof of the Projection and the Selection Theorems. The proof of these theorems depends on the theory of Suslin sets. The theory of Suslin sets describes the images of measurable sets under continuous mappings[1] and the theory depends on the delicate interaction of the measurable and topological structure of complete separable metric spaces. We will not discuss this theory in detail we are merely trying to give a simple proof of the aforementioned theorems. The most remarkable phenomena one should follow below is the role of the completeness of the underlying σ-algebra, that is the role of the subsets of the measure zero sets.

1 The definition of the Suslin sets

Definition 1 *If (X, τ) is a topological space then $\mathcal{B}(X)$ denotes the Borel σ-algebra generated by (X, τ). In these notes $\mathcal{F}(X)$ will denote the set of closed sets.*

Lemma 2 *If the topological space (X, τ) is metrizable then $\mathcal{B}(X)$ is the smallest family of sets containing the open sets and closed under countable union and intersection. The same is true for the closed sets of X.*

Proof: Let $\mathcal{A}$ denote the family of sets in the lemma. By the definition of the Borel σ-algebra $\mathcal{A} \subseteq \mathcal{B}(X)$. As (X, τ) is metrizable every closed set is a countable intersection of open sets, so $\mathcal{F}(X) \subseteq \mathcal{A}$. Let

$$\mathcal{A}_0 \overset{\circ}{=} \{A : A \in \mathcal{A} \text{ and } A^c \in \mathcal{A}\}.$$

As $\mathcal{A}$ contains the closed and the open sets the open sets are in $\mathcal{A}_0$. $\mathcal{A}$ is closed under the countable union and intersection therefore $\mathcal{A}_0$ is also

[1]The inverse images of open sets under continuous mappings are open. The question is what we know about the images of open sets. Are they measurable at all?

closed under the countable union and intersection, so by the definition of $\mathcal{A}$ obviously $\mathcal{A}_0 \subseteq \mathcal{A}$, that is $\mathcal{A}_0 = \mathcal{A}$. Hence $\mathcal{A}$ is closed under complementation as well, so $\mathcal{A}$ is a σ-algebra. Therefore $\mathcal{B}(X) \subseteq \mathcal{A}$, hence $\mathcal{A} = \mathcal{B}(X)$. To prove the second part of the lemma one should remark that in a metrizable space every open set is a countable union of closed sets.

□

Definition 3 *A topological space (X, τ) is called a Polish space if there is a metric $d : X \times X \to \mathbb{R}_+$ generating τ such that (X, d) is a complete separable metric space.*

Obviously a closed subset of a Polish space is a Polish space and the topological product of countable many Polish spaces is a Polish space.

Definition 4 *Let (X, τ) be a topological space. An $S \subseteq X$ is a Suslin set in X if there is a Polish space P and a continuous mapping $h : P \to X$ such that $S = h(P)$. The topological space (X, τ) is a Suslin space if X is a Suslin subset of the topological space (X, τ). We shall denote the set of Suslin subsets of X by $\mathcal{S}(X)$.*

Lemma 5 *The following properties of the Suslin sets are obvious:*

1. *If X and Y are topological spaces, $f : X \to Y$ is continuous and S is a Suslin set in X then $f(S)$ is a Suslin set in Y.*

2. *If (S_n) are countable many Suslin spaces then the topological product $\Pi_n S_n$ is also a Suslin space.*

3. *Every Polish space is a Suslin space and a closed subset of a Polish space is a Suslin set.*

The next property of the Suslin sets is crucial.

Lemma 6 *The Suslin sets of a topological space are closed under the countable union and intersection[2].*

Proof: Let (S_n) be a sequence of Suslin sets in a topological space X. Let T_n be a Polish space and let $f_n : T_n \to X$ be continuous with $f(T_n) = S_n$. Let $T \doteq \Pi_n T_n$ be the topological product of the Polish spaces T_n. Obviously T is a Polish space. Observe that $\mathbb{N}$ with the discrete topology is also a Polish space so $\mathbb{N} \times T$ is a Polish space. If

$$h(n, (t_1, t_2, t_3, \ldots)) \doteq h_n(t_n)$$

[2] But not under the complementation!

then h is trivially continuous so $\cup_n A_n = h(\mathbb{N} \times T)$ is a Suslin set in X.

$$Z \mathrel{\hat{=}} \{(n, (t_1, t_2, t_3, \ldots)) \in \mathbb{N} \times T : h_1(t_1) = h_2(t_2) = h_3(t_3) = \cdots\}$$

is a closed subset of $\mathbb{N} \times T$ hence it is a Polish space. So $\cap_n A_n = h(Z)$ is a Suslin set in X.

□

Every closed subset of a Polish space is a Suslin set. As every open set in a metric space is a countable union of closed sets it is clear that every open set of a Polish space is also a Suslin set. One can prove more:

Lemma 7 *If X is a Polish space then $\mathcal{B}(X) \subseteq \mathcal{S}(X)$. The same is true if X is a Suslin space.*

Proof: $\mathcal{S}(X)$ is closed under the countable union and intersection. If X is a Polish space then $\mathcal{F}(X) \subseteq \mathcal{S}(X)$, so by Lemma 2 $\mathcal{B}(X) \subseteq \mathcal{S}$. If X is a Suslin space then $X = h(T)$, where T is a Polish space. If $A \in \mathcal{B}(X)$ then $h^{-1}(A) \in \mathcal{B}(T) \subseteq \mathcal{S}(T)$, hence as h is continuous $A = h(h^{-1}(A)) \in \mathcal{S}(X)$ by Lemma 5.

□

Example 8 *If X is a separable Banach space and if $E, F \in \mathcal{B}(X)$, then $E + F \in \mathcal{S}(X)$.*

$Y \mathrel{\hat{=}} X \times X$ is a Polish space and $G \mathrel{\hat{=}} E \times F \in \mathcal{B}(Y) \subseteq \mathcal{S}(Y)$. The function $h(x, y) \mathrel{\hat{=}} x + y$ is continuous so $h(G) \in \mathcal{S}(X)$.

□

Example 9 *If X and Y are Suslin spaces and $B \in \mathcal{B}(X \times Y)$ then $\mathrm{pr}_X(B) \in \mathcal{S}(X)$.*

It is sufficient to recall that the projection is continuous.

□

2 The measurability of Suslin sets

Lemma 10 *Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and let*

$$\mu^*(A) \mathrel{\hat{=}} \inf\{\mu(B) : A \subseteq B \in \mathcal{B}\}$$

be the outer measure generated by μ. If $A_n \nearrow A$ then

$$\lim_{n \to +\infty} \mu^*(A_n) = \mu^*(A).$$

Proof: If $\mu^*(A_n) = +\infty$ for some n then the relation is obvious, therefore one can assume that $\mu^*(A_n) < +\infty$ for every n. As $\mathcal{B}$ is a σ-algebra by the definition of μ^* for every n there is a $B_n \in \mathcal{B}$ such that $A_n \subseteq B_n$ and $\mu^*(A_n) = \mu(B_n)$. For every $1 \leq k \leq n$ obviously $A_k \subseteq B_k \cap B_n \subseteq B_k$ hence

$$\mu(B_k) = \mu^*(A_k) \leq \mu(B_k \cap B_n) \leq \mu(B_k).$$

Therefore

$$\mu(B_k \cap B_n) = \mu(B_k).$$

As μ is additive

$$\mu(B_k) = \mu(B_k \cap B_n) + \mu(B_k \setminus B_n)$$

and as $\mu(B_k) < +\infty$ obviously $\mu(B_k \setminus B_n) = 0$. Hence for every n

$$\mu(\cup_{k=1}^n B_k) = \mu(B_n).$$

As μ^* is increasing

$$\lim_{n\to+\infty} \mu(B_n) = \lim_{n\to+\infty} \mu^*(A_n) \leq \mu^*(\cup_n A_n) \leq$$
$$\leq \mu(\cup_n B_n) = \lim_{n\to+\infty} \mu(\cup_{k=1}^n B_k) = \lim_{n\to+\infty} \mu(B_n).$$

Hence the second relation above is an equality which proves the lemma.

□

From now on let X be a metric space and let $(X, \mathcal{B}(X), \mu)$ be a finite measure space. Let $\mu^* : \mathcal{P}(X) \to \mathbb{R}_+$ be the outer measure generated by μ. The following proposition shows the importance of Polish spaces in the theory of Suslin sets.

Definition 11 *Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. By definition the μ^*-measurability of a set $S \subseteq \Omega$ means that there are $B_1, B_2 \in \mathcal{B}$ such that $B_1 \subseteq S \subseteq B_2$ and $\mu(B_2 \setminus B_1) = 0$.*

Proposition 12 *If X is a metric space then every $S \in \mathcal{S}(X)$ is μ^*-measurable.*

Proof: Let $f(T) = S$ where T is a Polish space and $f : T \to X$ is continuous. As T is a Polish space $T = \cup_j T_j$, where $\delta(T_j) \leq 1$ and every T_j is closed. (One can use the closed balls with radius $1/2$ around points of a dense, countable subset of T.) Let

$$A_n \doteq f\left(\cup_{j=1}^n T_j\right).$$

As every T_j is a Polish space it has a decomposition $T_j \circeq \cup_k T_{jk}$, where T_{jk} are closed and $\delta(T_{jk}) \leq 1/2$. Let

$$A_{jn} \circeq f\left(\cup_{k=1}^{n} T_{jk}\right).$$

One can continue the procedure and construct a sequence

$$A_{n_1,n_2,\ldots,n} \circeq f\left(\cup_{k=1}^{n} T_{n_1,n_2,\ldots,k}\right)$$

where $\delta\left(T_{n_1,n_2,\ldots,n_k}\right) \leq 1/2^k$. Obviously for every k and $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$

$$A_{n_1,n_2,\ldots,n_k,n} \nearrow A_{n_1,n_2,\ldots,n_k}.$$

Let $\varepsilon > 0$. As $A_n \nearrow S$ by the previous lemma there is an index $n_1 \in \mathbb{N}$ such that

$$\mu^*\left(A_{n_1}\right) > \mu^*(S) - \frac{\varepsilon}{2}.$$

Now assume that we have already defined the indexes $n_1, n_2, \ldots, n_k$. As $A_{n_1,n_2,\ldots,n_k,n} \nearrow A_{n_1,n_2,\ldots,n_k}$ there is an index $n_{k+1} \in \mathbb{N}$ such that

$$\mu^*\left(A_{n_1,n_2,\ldots,n_k,n_{k+1}}\right) > \mu^*\left(A_{n_1,n_2,\ldots,n_k}\right) - \frac{\varepsilon}{2^{k+1}}.$$

The sequence $k \mapsto A_{n_1,n_2,\ldots,n_k}$ is obviously decreasing and for every $k \in \mathbb{N}$

$$\mu^*\left(A_{n_1,n_2,\ldots,n_k}\right) > \mu^*(S) - \sum_{j=1}^{k} \frac{\varepsilon}{2^j} > \mu^*(S) - \varepsilon.$$

For every $k \in \mathbb{N}$ let

$$H_k \circeq \operatorname{cl}\left(A_{n_1,n_2,\ldots,n_k}\right) \in \mathcal{F}(X)$$

and let

$$H^{(\varepsilon)} \circeq \cap_k H_k \in \mathcal{F}(X) \subseteq \mathcal{B}(X).$$

The sequence (H_k) is decreasing and as μ is finite

$$\mu\left(H^{(\varepsilon)}\right) = \lim_{k\to+\infty} \mu(H_k) \geq \lim_{k\to+\infty} \mu^*\left(A_{n_1,n_2,\ldots,n_k}\right) \geq \mu^*(S) - \varepsilon.$$

We show that $H^{(\varepsilon)} \subseteq S$. Let $x \in H^{(\varepsilon)}$. This means that $x \in H_k$ for every k. By the definition of H_1

$$B(x,1) \cap A_{n_1} \neq \emptyset.$$

This implies that

$$f(T_n) \cap B(x,1) \neq \emptyset$$

for certain sets T_n with $n \leq n_1$. Now by the definition of H_2

$$B(x, 1/2) \cap A_{n_1 n_2} \neq \emptyset$$

which implies that for certain indexes $n \leq n_1$ and $m \leq n_2$

$$f(T_{nm}) \cap B(x, 1/2) \neq \emptyset.$$

One can continue the procedure for all $n_1, n_2 \ldots, n_k$. As the number of sets $T_{m_1, m_2, \ldots, m_k}$ in the construction is infinite there is an index $m_1 \leq n_1$ such that T_{m_1} contains infinitely many of the selected T-sets. As the number of the sets $T_{m_1 n}$ is also finite there is an index $m_2 \leq n_2$ such that $T_{m_1 m_2}$ contains infinitely many T-sets. Therefore one can construct a sequence $s_k \in T_{m_1, m_2, \ldots, m_k}$ such that

$$d(x, f(s_k)) \leq 1/2^k.$$

As $T_{m_1, m_2, \ldots, m_{k+1}} \subseteq T_{m_1, m_2, \ldots, m_k}$ by the construction $d(s_k, s_{k+1}) < 1/2^k$, therefore $(s_k) \subseteq T$ is a Cauchy sequence. As T is complete there is an $s \in T$ such that $f(s_n) \to f(s)$ which means that $x = f(s) \in S$. Therefore $H^{(\varepsilon)} \subseteq S$. Let now

$$H_* \circeq \cup_n H_{1/n}.$$

Obviously $H_* \in \mathcal{B}(X)$ and $H_* \subseteq S$ and also for every n

$$\mu(H_*) \geq \mu\left(H^{(\varepsilon)}\right) \geq \mu^*(S) - \frac{1}{n}$$

which means that $\mu(H_*) = \mu^*(S)$. On the other hand by the definition of the outer measure there is a set $K_* \in \mathcal{B}(X)$ such that $S \subseteq K^*$ and

$$\mu(K_*) = \mu^*(S) = \mu(H_*).$$

Hence $K_* \setminus H_*$ has μ-measure zero and it contains the set $S \setminus H_*$. This implies that $S \setminus H_*$ has μ^*-measure zero and since H_* is μ-measurable S will be also μ^*-measurable.

□

Definition 13 *The measurable space* $(\Omega, \mathcal{A})$ *is called complete if for every* $E \subseteq \Omega$, $E \notin \mathcal{A}$ *there is a finite measure* $\mu : \mathcal{A} \to \mathbb{R}_+$ *such that* E *is not* μ^**-measurable.*

Example 14 *If* $(\Omega, \mathcal{A}, \mu)$ *is a finite or* σ*-finite, complete measure space*[3] *then* $(\Omega, \mathcal{A})$ *is a complete measurable space.* $\mathbb{R}^n$ *with the Lebesgue measurable sets is a complete measure space.*

[3] The completeness of a measure space means that all subsets of measure zero sets are measurable.

Proposition 15 *Let $(\Omega, \mathcal{A})$ be a complete measurable space, X be a metric space and let $f : \Omega \to X$ be an $\mathcal{A}$-measurable mapping*[4]. *Then*

$$f^{-1}(S) \in \mathcal{A}$$

for every $S \in \mathcal{S}(X)$ that is the inverse images of the Suslin sets are measurable.

Proof: Assume that $f^{-1}(S) \notin \mathcal{A}$. Then by definition there is a finite measure $\mu : \mathcal{A} \to \mathbb{R}_+$ such that $f^{-1}(S)$ is not μ^*-measurable. If

$$\nu(A) \circeq \mu\left(f^{-1}(A)\right)$$

then $(X, \mathcal{B}(X), \nu)$ is a finite measure space. Let ν^* be the outer measure of ν. By the previous proposition S is ν^*-measurable, hence S has the decomposition $S \circeq K \cup L$ where $K \in \mathcal{B}(X)$ and $\nu^*(L) = 0$. Hence there is an $F \in \mathcal{B}(X)$ such that $L \subseteq F$ and $\nu(F) = 0$. As f measurable $f^{-1}(K) \in \mathcal{A}$ and by the definition of ν $f^{-1}(F) \in \mathcal{A}$ has μ-measure zero. Hence $f^{-1}(L) \subseteq f^{-1}(F)$ is μ^*-measurable. Therefore

$$f^{-1}(S) = f^{-1}(K \cup L) = f^{-1}(K) \cup f^{-1}(L)$$

is μ^*-measurable. This is a contradiction therefore $f^{-1}(S) \in \mathcal{A}$.

□

Example 16 *Every $S \in \mathcal{S}(\mathbb{R})$ is Lebesgue measurable.*

With $f(x) \circeq x$ it is trivial from the just proved proposition.

□

Example 17 *Let X be a Suslin space and let $B \in \mathcal{B}(\mathbb{R} \times X)$. Then* $\mathrm{pr}_{\mathbb{R}}(B)$ *is Lebesgue measurable.*

$\mathrm{pr}_{\mathbb{R}}(B)$ is a Suslin subset of $\mathbb{R}$ so it is Lebesgue measurable.

□

Example 18 *If $E, F \in \mathcal{B}(\mathbb{R})$, then $E + F$ is Lebesgue measurable.*

[4] That is $f^{-1}(\mathcal{B}(X)) \subseteq \mathcal{A}$.

3 Measurable Projection Theorem

Lemma 19 *For every* $n \in \mathbb{N}$

$$H_n \circeq \left\{ \sum_{k=1}^{+\infty} \frac{a(k)}{4^k} : a \in \{0,1\}^{\mathbb{N}} \text{ and } a(n) = 1 \right\}$$

is a compact subset of $\mathbb{R}$. *For every* $a \in \{0,1\}^{\mathbb{N}}$

$$\sum_{k=1}^{+\infty} \frac{a(k)}{4^k} \in H_n$$

if and only if $a(n) = 1$.

Proof: The topological product $\{0,1\}^{\mathbb{N}}$ is compact and

$$M_n \circeq \left\{ a \in \{0,1\}^{\mathbb{N}} : a(n) = 1 \right\}$$

is a closed subset of this product. The function $f(a) \circeq \sum_{k=1}^{+\infty} \frac{a(k)}{4^k}$ is a uniformly convergent limit of continuous functions so it is continuous. Hence $H_n = f(M_n)$ is compact. f is injective[5] so $f(a) \in H_n = f(M_n)$ is equivalent to the condition $a \in M_n$.

□

Lemma 20 *Let* $(\Omega, \mathcal{A})$ *be a measurable space and let* X *be a topological space. For every* $C \in \mathcal{A} \times \mathcal{B}(X)$ *there is a set* $D \in \mathcal{B}(\mathbb{R} \times X)$ *and an* $\mathcal{A}$*-measurable function* $g : \Omega \to \mathbb{R}$ *such that*

$$C = \{(\omega, x) : (g(\omega), x) \in D\}.$$

Proof: As C is in the σ-algebra generated by the measurable rectangles of $\mathcal{A}$ and $\mathcal{B}(X)$ it is also in a σ-algebra generated by countable many measurable rectangle, therefore there are sets $(A_n) \subseteq \mathcal{A}$ and $(B_n) \subseteq \mathcal{B}(X)$ such that

$$C \in \mathcal{G} \circeq \sigma(\{A_n \times B_n : n \in \mathbb{N}\}).$$

Let

$$g(\omega) \circeq \sum_{k=1}^{+\infty} \frac{1}{4^k} \cdot \chi_{A_k}(\omega).$$

[5] The simplest way to show this just to observe that if a sequence x is lexicographically smaller than y then $f(x) < f(y)$. To show this one should use that in the definition of f one has 4 and not 2.

By the previous lemma for every $n \in \mathbb{N}$ and $\omega \in \Omega$

$$g(\omega) \in H_n \Longleftrightarrow \chi_{A_n}(\omega) = 1 \Longleftrightarrow \omega \in A_n,$$

that $g^{-1}(H_n) = A_n$, where $H_n \subseteq \mathbb{R}$ are the compact sets in the previous lemma. As g is a sum of $\mathcal{A}$-measurable functions it is $\mathcal{A}$-measurable. Let

$$h(\omega, x) \mathrel{\hat{=}} (g(\omega), x),$$

and let

$$\mathcal{H} \mathrel{\hat{=}} \left\{h^{-1}(D) : D \in \mathcal{B}(\mathbb{R} \times X)\right\}.$$

$\mathcal{H}$ is obviously a σ-algebra in $\Omega \times X$. As $g^{-1}(H_n) = A_n$ for every n by the definition of h

$$h^{-1}(H_n \times B_n) = g^{-1}(H_n) \times B_n = A_n \times B_n.$$

As obviously $D_n \mathrel{\hat{=}} H_n \times B_n \in \mathcal{B}(\mathbb{R} \times X)$ by the definition of $\mathcal{H}$

$$A_n \times B_n \in \mathcal{H}.$$

$\mathcal{H}$ is a σ-algebra and $\mathcal{G}$ is the smallest σ-algebra generated by the rectangles $A_n \times B_n$ therefore $\mathcal{G} \subseteq \mathcal{H}$. Hence $C \in \mathcal{G} \subseteq \mathcal{H}$, that is by the definition of $\mathcal{H}$ there is a set $D \in \mathcal{B}(\mathbb{R} \times X)$ with $C = h^{-1}(D)$. By the definition of h this implies the lemma.

□

Theorem 21 (Measurable Projection Theorem) *Let $(\Omega, \mathcal{A})$ be a complete measurable space and let X be a Suslin space. If $C \in \mathcal{A} \times \mathcal{B}(X)$ then*

$$\mathrm{pr}_\Omega(C) \in \mathcal{A}.$$

Proof: By the previous lemma there is a $D \in \mathcal{B}(\mathbb{R} \times X)$ and an $\mathcal{A}$-measurable function $g : \Omega \to \mathbb{R}$ such that

$$C = \{(\omega, x) : (g(\omega), x) \in D\}.$$

From this trivially

$$\mathrm{pr}_\Omega(C) = g^{-1}(\mathrm{pr}_\mathbb{R}(D)).$$

As $\mathbb{R} \times X$ is a Suslin space $D \in \mathcal{B}(\mathbb{R} \times X)$ is a Suslin set[6] so $S \mathrel{\hat{=}} \mathrm{pr}_\mathbb{R}(D)$ is a continuous image of a Suslin set so S is a Suslin set in $\mathbb{R}$. But as g is $\mathcal{A}$-measurable[7]

$$\mathrm{pr}_\Omega(C) = g^{-1}(S) \in \mathcal{A}.$$

□

[6] See: Lemma 7.
[7] See: Proposition 15.

4 Measurable Selection Theorem

The topological version of the Measurable Selection Theorem is the following:

Lemma 22 (Kuratowski–Ryll-Nardzewski) *Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a Polish space. Let $F : \Omega \to \mathcal{F}(X)$ be a point to set mapping and assume that*

1. *$F(\omega) \neq \emptyset$ for every $\omega \in \Omega$ and*

2. *for every open set $G \subseteq X$*

$$F^{-}(G) \circeq \{\omega \in \Omega : F(\omega) \cap G \neq \emptyset\} \in \mathcal{A}.$$

Then F has a measurable selection $f : \Omega \to X$ that is there is an $\mathcal{A}$-measurable function f such that $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.

Proof: In order to simplify the notation one may assume that $\delta(X) \leq 1$. As X is complete for the proof of the the theorem it is sufficient to construct a sequence of measurable mappings (f_n) with the properties

1. $d(f_n(\omega), F(\omega)) < 1/2^n$

2. $d(f_n(\omega), f_{n-1}(\omega)) < 1/2^{n-1}$.

Let $Q = (q_k)$ be a countable dense subset of X. Let be choosen $f_0 \equiv q \in Q$ arbitrarily. As $\delta(X) \leq 1$ conditions 1 and 2 hold. Assume that f_{n-1} has already been constructed. Let

$$\begin{aligned} C_k^n &\circeq \left\{\omega : d(q_k, F(\omega)) < \frac{1}{2^n}\right\}, \\ D_k^n &\circeq \left\{\omega : d(q_k, f_{n-1}(\omega)) < \frac{1}{2^{n-1}}\right\}, \\ A_k^n &\circeq C_k^n \cap D_k^n. \end{aligned}$$

Choose $\omega \in \Omega$. By the construction of f_{n-1} there is an $x \in F(\omega)$ such that $d(x, f_{n-1}(\omega)) < 1/2^{n-1}$. As Q is dense in X there is a $q_k \in Q$ such that $d(q_k, x) < 1/2^n$ and

$$d(q_k, f_{n-1}(\omega)) \leq d(q_k, x) + d(x, f_{n-1}(\omega)) < \frac{1}{2^{n-1}}.$$

This means that $\omega \in A_k^n$. Therefore $\Omega = \cup_k A_k^n$ for any n. If

$$B_k^n \circeq \left\{x : d(q_k, x) < \frac{1}{2^n}\right\}$$

then

$$C_k^n = \{\omega : F(\omega) \cap B_k^n \neq \emptyset\},$$
$$D_k^n = f_{n-1}^{-1}\left(B_k^{n-1}\right).$$

As f_{n-1} is measurable $D_k^n \in \mathcal{A}$. By the conditions of the lemma $C_k^n \in \mathcal{A}$ as well, so $A_k^n \in \mathcal{A}$. Let

$$f_n(\omega) \circeq q_k, \quad \omega \in A_k^n \setminus \cup_{r=1}^{k-1} A_r^n \in \mathcal{A}.$$

For any open set G

$$f_n^{-1}(G) = \cup_{q_k \in G} f_n^{-1}(q_k) \in \mathcal{A}$$

so f_n is measurable. It is also clear from the construction that 1. and 2. hold. $(f_n(\omega))$ is a Cauchy sequence and as X is complete $f_n(\omega) \to f(\omega)$ for some $f(\omega)$ and as X is a separable metric space f is also measurable. As $F(\omega)$ is closed it is obvious that

$$f(\omega) \in \operatorname{cl}(F(\omega)) = F(\omega)$$

for every ω. Hence f is a measurable selection of F.

□

Theorem 23 (Measurable Selection Theorem) *Let X be a Suslin space and let $(\Omega, \mathcal{A})$ be a complete measurable space. Let $F : \Omega \to \mathcal{P}(X)$ be a point to set mapping and let assume that*

1. *$F(\omega) \neq \emptyset$ for every $\omega \in \Omega$ and*
2. $\operatorname{graph} F \in \mathcal{A} \otimes \mathcal{B}(X)$.

Then F has a measurable selection $f : \Omega \to X$ that is there is an $\mathcal{A}$-measurable function f such that $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.

Proof: As in the proof of the Projection Theorem there is a set $D \in \mathcal{B}(\mathbb{R} \times X)$ and an $\mathcal{A}$-measurable function $g : \Omega \to \mathbb{R}$ such that

$$\operatorname{graph} F = \{(\omega, x) : (g(\omega), x) \in D\}.$$

As $D \in \mathcal{B}(\mathbb{R} \times X)$, it is a Suslin set in $\mathbb{R} \times X$, hence there is a Polish space Y and a continuous function $h : Y \to \mathbb{R} \times X$ such that $h(Y) = D$. Let

$$\Gamma(\omega) \circeq h^{-1}(\{g(\omega)\} \times X).$$

As $F(\omega) \neq \emptyset$ there is an x such that $(\omega, x) \in \operatorname{graph} F$, which implies that

$$(g(\omega), x) \in D = h(Y).$$

Hence for every ω there is an $y \in Y$ such that $h(y) = (g(\omega), x)$. By the definition of Γ this means that $y \in \Gamma(\omega)$. Hence $\Gamma(\omega) \neq \emptyset$ for every $\omega \in \Omega$. As h is continuous $\Gamma(\omega) \in \mathcal{F}(Y)$. As Y is a Polish space every open set $G \subseteq Y$ is a Suslin set. As h is continuos $\mathrm{pr}_{\mathbb{R}}(h(G))$ is a Suslin set in $\mathbb{R}$. As g is $\mathcal{A}$-measurable

$$\Gamma^{-}(G) = g^{-1}(\mathrm{pr}_{\mathbb{R}}(h(G))) \in \mathcal{A}.$$

Hence Γ satisfies the Kuratowski–Ryll-Nardzewski selection theorem. Let $\varphi : \Omega \to Y$ be the $\mathcal{A}$-measurable selection of Γ. For very $\omega \in \Omega$

$$h(\varphi(\omega)) \in (\{g(\omega)\} \times X) \cap D.$$

Hence $\mathrm{pr}_{\mathbb{R}}(h(\varphi(\omega))) = g(\omega)$ for every ω. Let $f(\omega) \circeq \mathrm{pr}_X(h(\varphi(\omega)))$. Then $f : \Omega \to X$ is $\mathcal{A}$-measurable and for every $\omega \in \Omega$

$$(g(\omega), f(\omega)) = (\mathrm{pr}_{\mathbb{R}}(h(\varphi(\omega))), \mathrm{pr}_X(h(\varphi(\omega)))) = h(\varphi(\omega)) \in D.$$

By the construction of D and g this implies that $(\omega, f(\omega)) \in \operatorname{graph} F$, that is f is an $\mathcal{A}$-measurable selection of F.

$\square$